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**Truth Degrees, Relevance, and  
Paraconsistency**

## What is a Substructural Propositional Logic?

- Each connective of the language has a **classical** counterpart.
- The consequence relation can be defined using a **Gentzen-type system** having two types of rules:
  - Structural Rules:** Do not involve any specific connective
  - Logical Rules:** Each involving exactly one connective.
- The set of rules for each of the connectives is adequate for its classical counterpart.
- *The cut-elimination theorem obtains*

## Some Structural Rules

**Weakening:**

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

**Contraction:**

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

**Expansion:**

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi, \varphi, \Gamma \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi, \varphi}$$

**Mingle:**

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

## Expansion=Mingle?

- **Mingle+Contraction** entail **Expansion**:

$$\frac{\frac{\varphi, \Gamma \Rightarrow \Delta \quad \varphi, \Gamma \Rightarrow \Delta}{\varphi, \varphi, \Gamma, \Gamma \Rightarrow \Delta, \Delta} \text{ (Mingle)}}{\varphi, \varphi, \Gamma \Rightarrow \Delta} \text{ (Contractions)}$$

- **Expansion+1** entail **Mingle**:

$$\frac{\frac{\frac{\frac{\mathbf{1} \Rightarrow \mathbf{1}}{\mathbf{1} \Rightarrow \mathbf{1}, \mathbf{1}} \text{ (Exp)}}{\mathbf{1}, \Gamma_1 \Rightarrow \Delta_1, \mathbf{1}} \text{ (Cut)}}{\mathbf{1}, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \text{ (Cut)}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \text{ (Cut)}}{\frac{\Gamma_1 \Rightarrow \Delta_1}{\mathbf{1}, \Gamma_1 \Rightarrow \Delta_1} \text{ (1} \Rightarrow \text{)}} \frac{\frac{\Gamma_2 \Rightarrow \Delta_2}{\mathbf{1}, \Gamma_2 \Rightarrow \Delta_2} \text{ (1} \Rightarrow \text{)}}{\mathbf{1}, \Gamma_2 \Rightarrow \Delta_2} \text{ (Cut)}}{\mathbf{1}} \text{ (Cut)}$$

## Blame it on $\&$

- Expansion +  $\{\rightarrow, \&\}$  entail Mingle:

Replace **1** by a sentence of the form

$$(\varphi_1 \rightarrow \varphi_1) \& \dots \& (\varphi_n \rightarrow \varphi_n)$$

- Expansion +  $\&$  suffice for deriving  $\Rightarrow \varphi, \psi$  from  $\Rightarrow \varphi$  and  $\Rightarrow \psi$ :

$$\begin{array}{c}
 \begin{array}{c} \Rightarrow \varphi \quad \Rightarrow \psi \\ \hline \Rightarrow \varphi \& \psi \end{array} \quad \begin{array}{c} \varphi \Rightarrow \varphi \\ \hline \varphi \& \psi \Rightarrow \varphi \end{array} \quad \begin{array}{c} \psi \Rightarrow \psi \\ \hline \varphi \& \psi \Rightarrow \psi \end{array} \\
 \hline
 \begin{array}{c} \Rightarrow \varphi, \varphi \& \psi \quad \quad \quad \varphi \& \psi \Rightarrow \psi \\ \hline \Rightarrow \varphi, \psi \end{array}
 \end{array}$$

## The Substructural Systems $R_m$ and $RMI_m$

**Structural Rules:** Identity, Cut, Contraction, **Expansion**

**Logical Rules:**

$$(\neg \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg \varphi, \Gamma \Rightarrow \Delta} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \varphi} \qquad (\Rightarrow \neg)$$

$$(\otimes \Rightarrow) \quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \otimes \psi \Rightarrow \Delta} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \quad \Gamma_2 \Rightarrow \Delta_2, \psi}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi \otimes \psi} \qquad (\Rightarrow \otimes)$$

$$(\rightarrow \Rightarrow) \quad \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \quad \psi, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \rightarrow \psi \Rightarrow \Delta_1, \Delta_2} \qquad \frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi} \qquad (\Rightarrow \rightarrow)$$

## The Basic Hilbert-type System $HR_m$

### Axioms

(I)	$\varphi \rightarrow \varphi$	(Identity)
(T)	$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \theta) \rightarrow (\varphi \rightarrow \theta))$	(Transitivity)
(P)	$(\varphi \rightarrow (\psi \rightarrow \theta)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \theta))$	(Permutation)
(R1)	$(\varphi \rightarrow (\psi \rightarrow \theta)) \rightarrow (\varphi \otimes \psi \rightarrow \theta)$	(Residuation)
(R2)	$(\varphi \otimes \psi \rightarrow \theta) \rightarrow (\varphi \rightarrow (\psi \rightarrow \theta))$	
(C)	$\varphi \rightarrow \varphi \otimes \varphi$	(Contraction)
(N1)	$(\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi)$	(Contraposition)
(N2)	$\neg\neg\varphi \rightarrow \varphi$	(Double Negation)

### Rule of inference

$$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi}$$

## Properties of $RMI_m$

- An equivalent Hilbert-type system  $HRMI_m$  is obtained by adding to  $HR_m$  the **mingle axiom**  $\varphi \otimes \varphi \rightarrow \varphi$ :
  1. A sequent is provable in  $RMI_m$  iff its translation is provable in  $HRMI_m$ . **In particular:**  $\vdash_{RMI_m} \varphi \iff \vdash_{HRMI_m} \varphi$ .
  2.  $\mathcal{T} \vdash_{HRMI_m} \varphi$  iff  $\vdash_{RMI_m} \Gamma \Rightarrow \varphi$  for some finite  $\Gamma \subseteq \mathcal{T}$ .
- **Relevant Deduction Theorem:**  $\mathcal{T}, \varphi \vdash_{RMI_m} \psi$  iff either  $\mathcal{T} \vdash_{RMI_m} \psi$  or  $\mathcal{T} \vdash_{RMI_m} \varphi \rightarrow \psi$ .
- **Variable-sharing:** If  $\vdash_{RMI_m} \varphi \rightarrow \psi$  then  $\varphi$  and  $\psi$  share a variable.
- **$\otimes$  as a conjunction:**  $\vdash_{RMI_m} \varphi \otimes \psi$  iff  $\vdash_{RMI_m} \varphi$  and  $\vdash_{RMI_m} \psi$  (even though  $\varphi \otimes \psi \not\vdash_{RMI_m} \varphi$  and  $\varphi \otimes \psi \not\vdash_{RMI_m} \psi$ ).  
This is not true for  $R_m$ !

## Weakly Characteristic Semantics

The matrix  $\mathcal{A}_\omega = \langle A_\omega, \mathcal{D}_\omega, \mathcal{O}_\omega \rangle$  is defined as follows:

- $A_\omega = \{\mathbf{t}, \mathbf{f}, I_1, I_2, I_3, \dots\}$
- $\mathcal{D}_\omega = A_\omega - \{\mathbf{f}\} = \{\mathbf{t}, I_1, I_2, I_3, \dots\}$ .
- The operations in  $\mathcal{O}_\omega$  are the following:

$$\neg \mathbf{t} = \mathbf{f} \quad \neg \mathbf{f} = \mathbf{t} \quad \neg I_k = I_k \quad (k = 1, 2, \dots)$$

$$a \otimes b = \begin{cases} \mathbf{f} & a = \mathbf{f} \text{ or } b = \mathbf{f} \\ I_k & a = b = I_k \\ \mathbf{t} & \text{otherwise} \end{cases} \quad a \rightarrow b = \begin{cases} \mathbf{t} & a = \mathbf{f} \text{ or } b = \mathbf{t} \\ I_k & a = b = I_k \\ \mathbf{f} & \text{otherwise} \end{cases}$$

$x \otimes y = \min_{\preceq_{\mathcal{A}}} \{x, y\}$ , where  $\mathbf{f} \preceq_{\mathcal{A}} \mathbf{t} \preceq_{\mathcal{A}} I_i$ .

$x \rightarrow y \in \mathcal{D}_\omega$  iff  $x \leq y$ , where  $\mathbf{f} \leq I_i \leq \mathbf{t}$ .

## Weakly Characteristic Semantics (Continued)

**Weak soundness and completeness:**  $\vdash_{RMI_m} \varphi$  iff  $\vdash_{\mathcal{A}_\omega} \varphi$ .

**Corollary:**  $\vdash_{RMI_m} \Gamma \Rightarrow \Delta$  iff for every valuation  $v$  in  $\mathcal{A}_\omega$ , either  $v(\varphi) = \mathbf{f}$  for some  $\varphi \in \Gamma$ , or  $v(\varphi) = \mathbf{t}$  for some  $\varphi \in \Delta$ , or there exists  $k$  such that  $v(\varphi) = I_k$  for every  $\varphi \in \Gamma \cup \Delta$ .

**Scroggs' property:**  $RMI_m$  does not have a finite (weakly) characteristic matrix, although every proper extension of it does: to every proper extension  $\mathcal{L}$  of  $RMI_m$  there exists  $n \in \mathbb{N}$  such that  $\mathcal{A}_n$  is a weakly characteristic matrix for  $\mathcal{L}$ , where  $\mathcal{A}_n = \langle A_n, \mathcal{D}_n, \mathcal{O}_n \rangle$  is the submatrix of  $\mathcal{A}_\omega$  induced by  $A_n = \{\mathbf{t}, \mathbf{f}, I_1, \dots, I_n\}$ .

$$RM_m$$

$RM_m$  is obtained from  $RMI_m$  by adding the Mingle rule.

- An equivalent Hilbert-type system  $HRM_m$  is obtained by adding to  $HRMI_m$  as axiom either  $\neg(\varphi \rightarrow \psi) \rightarrow (\psi \rightarrow \varphi)$  or  $\neg(\varphi \rightarrow \varphi) \rightarrow (\psi \rightarrow \psi)$ .
- Relevant Deduction Theorem obtains.
- Variable-sharing fails, but  $RM_m$  is still **paraconsistent**.
- $RM_m$  is sound and **weakly** complete for  $\mathcal{A}_1$ .
- Not strongly complete:  $\varphi \otimes \psi \vdash_{\mathcal{A}_1} \varphi$  but  $\varphi \otimes \psi \not\vdash_{RM_m} \varphi$ .

## Strongly Characteristic Semantics for $RM_m$

- $S = (0, 1] \times \{\mathbf{t}, \mathbf{f}\} \cup \{\langle 0, I \rangle\}$   
 $deg(\langle a, x \rangle) = a \quad val(\langle a, x \rangle) = x$
- $\mathcal{D} = (0, 1] \times \{\mathbf{t}\} \cup \{\langle 0, I \rangle\} (= \{X \in S \mid val(X) \neq \mathbf{f}\})$ .
- $\neg \langle a, x \rangle = \langle a, \neg x \rangle$  ( $\neg \mathbf{t} = \mathbf{f}, \neg \mathbf{f} = \mathbf{t}, \neg I = I$ )  
 $deg(u \otimes v) = max\{deg(u), deg(v)\}$

$$val(u \otimes v) = \begin{cases} I & u = v \text{ and } val(u) = I \\ \mathbf{f} & deg(u) \geq deg(v) \text{ and } val(u) = \mathbf{f} \\ \mathbf{f} & deg(u) \leq deg(v) \text{ and } val(v) = \mathbf{f} \\ \mathbf{t} & \text{otherwise} \end{cases}$$

$$u \rightarrow v = \neg(u \otimes \neg v)$$

We denote the resulting structure by *SUG*.

## Strongly Characteristic Semantics for $RM_m$ (Continued)

- **Strong soundness and completeness:**  $\mathcal{T} \vdash_{RM_m} \varphi$  iff  $\mathcal{T} \vdash_{SUG} \varphi$ .
- Suppose that  $\Gamma \not\vdash_{RM_m} \varphi$ , and that  $\Gamma \cup \{\varphi\}$  involves at most  $n$  different propositional variables. Then there is a **submatrix**  $SUG(\Gamma, \varphi)$  of  $SUG$  such that  $SUG(\Gamma, \varphi)$  has at most  $2n$  elements, and there is a valuation in it which is a model of  $\Gamma$ , but not a model of  $\varphi$ .
- For  $n > 0$  there is a theory  $T_n$  in  $p_1, \dots, p_n$  such that  $T_n \not\vdash_{RM_m} p_1$ , but any model of  $T_n$  in  $SUG$  which is not a model of  $p_1$  involves **at least  $n$  different degrees**, and at least  $2n$  different elements of this matrix.

## Ordering *SUG* and *RM*

$X \preceq Y$  iff  $\text{deg}(X) > \text{deg}(Y)$  or  $\text{deg}(X) = \text{deg}(Y)$  and  $\text{val}(X) \preceq_{\mathcal{A}} \text{val}(Y)$ .

- $X \otimes Y = \min_{\preceq} \{X, Y\}$ .

$X \leq Y$  iff  $X \rightarrow Y \in \mathcal{D}$

- $\leq$  is a linear order on the elements of *SUG*.
- $\langle b, f \rangle < \langle a, f \rangle < \langle 0, I \rangle < \langle a, t \rangle < \langle b, t \rangle$  if  $0 < a < b$ .
- $X \rightarrow Y = \sup_{\leq} \{Z \mid X \otimes Z \leq Y\}$ .

Let  $X \wedge Y = \min_{\leq} \{X, Y\}$ ,  $X \vee Y = \max_{\leq} \{X, Y\}$ .

*RM* is the logic induced by *SUG* in the language with  $\wedge$  and  $\vee$ .

## $\mathcal{M}_0$ : An Isomorphic Version

- $[-1, 1]$  — the set of truth-values
- $[0, 1]$  — the set of designated values
- $\neg X = -X$

$$X \otimes Y = \min_{\preceq} \{X, Y\}$$

Where  $X \preceq Y$  if  $|X| > |Y|$  or  $|X| = |Y|$  and  $X \leq Y$ .

$$X \wedge Y = \min\{X, Y\}$$

$$X \vee Y = \max\{X, Y\}.$$

In the corresponding version of Łukasiewicz matrix, the set of designated values is  $\{1\}$ , while the interpretation of the intensional conjunction  $\otimes$  (or  $\&$ ) is given by  $X \otimes Y = \max\{X + Y - 1, -1\}$ .

## HR and HRM

Are obtained by respectively adding to  $HR_m$  and  $HRM_m$ :

**Rule:** Adjunction

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}$$

**Axioms:**

$$\text{(ConE1)} \quad \varphi \wedge \psi \rightarrow \varphi$$

$$\text{(ConE2)} \quad \varphi \wedge \psi \rightarrow \psi$$

$$\text{(ConI)} \quad (\theta \rightarrow \varphi) \wedge (\theta \rightarrow \psi) \rightarrow (\theta \rightarrow \varphi \wedge \psi)$$

$$\text{(DisI1)} \quad \varphi \rightarrow \varphi \vee \psi$$

$$\text{(DisI2)} \quad \psi \rightarrow \varphi \vee \psi$$

$$\text{(DisE)} \quad (\varphi \rightarrow \theta) \wedge (\psi \rightarrow \theta) \rightarrow (\varphi \vee \psi \rightarrow \theta)$$

$$\text{(Dist)} \quad \varphi \wedge (\psi \vee \theta) \rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \theta)$$

## Strongly Characteristic Semantics for $RMI_m$

- $SA = [0, 1] \times A_\omega$   
 $deg(\langle a, x \rangle) = a \quad val(\langle a, x \rangle) = x$
- $DA = [0, 1] \times \mathcal{D}_\omega \quad (= \{X \in S \mid val(X) \neq \mathbf{f}\})$ .
- $\neg \langle a, x \rangle = \langle a, \neg x \rangle$   
 $deg(u \otimes v) = max\{deg(u), deg(v)\}$   
 $val(u \otimes v) = \begin{cases} I_k & u = v \text{ and } val(u) = I_k \\ \mathbf{f} & deg(u) \leq deg(v) \text{ and } val(u) = \mathbf{f} \\ \mathbf{f} & deg(u) \geq deg(v) \text{ and } val(v) = \mathbf{f} \\ \mathbf{t} & \text{otherwise} \end{cases}$   
 $u \rightarrow v = \neg(u \otimes \neg v)$

We denote the resulting structure by  $SA$ .

## Strongly Characteristic Semantics (Continued)

- **Strong soundness and completeness:**  $\mathcal{T} \vdash_{RMI_m} \varphi$  iff  $\mathcal{T} \vdash_{\mathcal{SA}} \varphi$ .
- Suppose that  $\Gamma \not\vdash_{RMI_m} \varphi$ , and that  $\Gamma \cup \{\varphi\}$  involves at most  $n$  different propositional variables. Then there is a **submatrix**  $\mathcal{SA}(\Gamma, \varphi)$  of  $\mathcal{SA}$  such that  $\mathcal{SA}(\Gamma, \varphi)$  has at most  $3n - 1$  elements, and there is a valuation in it which is a model of  $\Gamma$ , but not a model of  $\varphi$ .
- For  $n > 0$  there is a theory  $T_n$  in  $p_1, \dots, p_n$  such that  $T_n \not\vdash_{RMI_m} p_1$ , but any model of  $T_n$  in  $\mathcal{SA}$  which is not a model of  $p_1$  involves **at least  $n$  different degrees**, and at least  $3n - 1$  different elements of  $\mathcal{SA}$ .

## Partially Ordering $\mathcal{SA}$

$\langle x, a \rangle \preceq_{\otimes} \langle y, b \rangle$  if either  $x > y$  &  $a \in \{\mathbf{t}, \mathbf{f}\}$ , or  $x = y$  and  $a \preceq_{\mathcal{A}} b$

- $\langle \mathcal{SA}, \preceq_{\otimes} \rangle$  is a lower semilattice.
- $X \otimes Y = \inf_{\preceq_{\otimes}} \{X, Y\}$ .

$X \leq Y$  iff  $X \rightarrow Y \in \mathcal{DA}$

- $\langle \mathcal{SA}, \leq \rangle$  is a lattice.
- $\langle b, f \rangle < \langle a, f \rangle < \langle a, I_k \rangle < \langle a, t \rangle < \langle b, t \rangle$  if  $a < b$ .
- $X \rightarrow Y = \sup_{\leq} \{Z \mid X \otimes Z \leq Y\}$ .

Let  $X \wedge Y = \inf_{\leq} \{X, Y\}$ ,  $X \vee Y = \sup_{\leq} \{X, Y\}$ .

**RMI** is the logic induced by  $\mathcal{SA}$  in the language with  $\wedge$  and  $\vee$ .

## HRMI

### Problems:

- $\mathcal{DA}$  is **not** closed under  $\wedge$ . Hence Adjunction is not valid.
- The lattice is **not** distributive. Hence (Dist) is not valid.

### Solution:

*HRMI* is obtained from *HRM* by replacing the Adjunction rule and axiom (Dist) with the following two rules:

$$\frac{\varphi \quad \psi \quad \varphi R \psi}{\varphi \wedge \psi} \qquad \frac{\psi R \theta}{\varphi \wedge (\psi \vee \theta) \rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \theta)}$$

Where  $\varphi R \psi$  is  $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$

(or  $\neg(\varphi \rightarrow \psi) \rightarrow (\psi \rightarrow \varphi)$  or  $\neg(\varphi \rightarrow \varphi) \rightarrow (\psi \rightarrow \psi)$ ).

**Theorem:** *HRMI* is strongly sound and complete for  $\mathcal{SA}$ .

## Another Semantic Approach

- $S = deg_1 \times \{\mathbf{t}, \mathbf{f}\} \cup deg_2 \times \{I\}$ , where:
  - $deg_1 \cap deg_2 = \emptyset$
  - $\langle Deg, \leq \rangle$  is an **inverse tree**, where  $Deg = deg_1 \cup deg_2$ .
  - If  $x \in deg_2$  then  $x$  is a  $\leq$ -minimal element of  $Deg$ .
- $\mathcal{D} = \{X \in S \mid val(X) \neq \mathbf{f}\}$
- $\neg \langle a, x \rangle = \langle a, \neg x \rangle$   
 $deg(u \otimes v) = \mathit{sup}_{\leq} \{deg(u), deg(v)\}$

$$val(u \otimes v) = \begin{cases} I & u = v \text{ and } val(u) = I \\ \mathbf{f} & deg(u) \geq deg(v) \text{ and } val(u) = \mathbf{f} \\ \mathbf{f} & deg(u) \leq deg(v) \text{ and } val(v) = \mathbf{f} \\ \mathbf{t} & \text{otherwise} \end{cases}$$

## RMI<sub>ma</sub> and RM<sub>ma</sub>

- Let  $RMI_{ma}$  be obtained from  $RMI_m$  by adding:

**Axioms:**

$$\perp, \Gamma \Rightarrow \Delta$$

$$\Gamma \Rightarrow \Delta, \top$$

**Rules:**

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

**Condition:** in  $(\Rightarrow \wedge)$  and  $(\vee \Rightarrow)$   $\Gamma \cup \Delta$  should not be empty!

- Let  $RM_{ma}$  be similarly obtained from  $RM_m$  (so  $RM_{ma}$  is  $RMI_{ma} + \text{mingle}$ ). Note that there is no need to remove the above restriction on  $(\Rightarrow \wedge)$  and  $(\vee \Rightarrow)$ .

## Properties of $RMI_{ma}$ and $RM_{ma}$

- $RMI_{ma}$  is **sound**, but **not complete** with respect to  $\mathcal{SA}$   
 $RM_{ma}$  is **sound**, but **not complete** with respect to  $\mathcal{SUG}$ .
- $RMI_{ma}$  ( $RM_{ma}$ ) is a **conservative** extension of  $RMI_m$  ( $RM_m$ ).
- $RMI_{ma}$  has the **variable-sharing** property.
- A **cut-free** formulation for  $RMI_{ma}$  is obtained by replacing the Expansion rules by the **Relevant Mingle** rules:

$$\frac{\varphi, \Gamma_1 \Rightarrow \Delta_1 \quad \varphi, \Gamma_2 \Rightarrow \Delta_2}{\varphi, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \quad \Gamma_2 \Rightarrow \Delta_2, \varphi}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi}$$

## Hypersequential Version of $RMI_{ma}$ and $RM_{ma}$

**Hypersequent:**  $s_1 \mid s_2 \mid \dots \mid s_n$  where  $s_1, \dots, s_n$  are sequents.

The hypersequential versions of  $RMI_{ma}$  and  $RM_{ma}$  include:

- The **hypersequential versions of the rules** of these systems. E.g.:

$$\frac{G \mid \Gamma, \varphi, \psi \Rightarrow \Delta}{G \mid \Gamma, \varphi \otimes \psi \Rightarrow \Delta} \quad \frac{G_1 \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \quad G_2 \mid \Gamma_2 \Rightarrow \Delta_2, \psi}{G_1, G_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi \otimes \psi}$$

- The basic **External structural rules**:

$$\frac{G \mid s \mid s}{G \mid s} \quad \frac{G}{G \mid s}$$

## The Relevant Hypersequential Calculi

*GRMI* - the logic of  $\mathcal{SA}$ : Add to  $RM_{ma}$  the **Splitting** rule:

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{G \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}$$

*GRM* - the logic of  $\mathcal{SUG}$ : Add the **Splitting** rule to  $RM_{ma}$ .

*GSRMI* - the logic of  $\mathcal{A}_\omega$ : Add to  $RM_{ma}$  **Strong Splitting**:

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{G \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2, \Gamma' \Rightarrow \Delta_2, \Delta'}$$

*GSRM* - the logic of  $\mathcal{A}_1$ : Add **Strong Splitting** to  $RM_{ma}$ .

An Example:  $\vdash_{GSRMI} \varphi \otimes \psi \Rightarrow \varphi \mid \Rightarrow \varphi$

(In general,  $\Gamma \vdash_{\mathcal{A}_\omega} \varphi$  iff  $\vdash_{GSRMI} \Gamma \Rightarrow \varphi \mid \Rightarrow \varphi$ ).

## Semantics and Strong Cut Elimination

Let  $G = \Gamma_1 \Rightarrow \Delta_1 | \cdots | \Gamma_m \Rightarrow \Delta_m$ .

- $\mathcal{F}_G = \bigcup_{i=1}^m (\Gamma_i \cup \Delta_i)$
- $\mathcal{M} \models G$  if for every valuation  $v$  in  $\mathcal{M}$  there is  $1 \leq i \leq m$  such that  $v \models \Gamma_i \cup \Delta_i$ .

**Theorem:** Let  $\mathcal{S} \cup \{G\}$  be a finite set of hypersequents in the language of  $RM$ . Then either there is a proof in  $GRM$  of  $G$  from  $\mathcal{S}$  in which all cuts are on formulas from  $\bigcup_{H \in \mathcal{S}} \mathcal{F}_H$ , or there is a model of  $\mathcal{S}$  in  $SUG$  which is not a model of  $G$ .

Similar Theorems obtain for  $GRMI$  and  $\mathcal{SA}$ ,  $GSRMI$  and  $\mathcal{A}_\omega$ , and  $GSRM$  and  $\mathcal{A}_1$ .

**Corollaries:** Strong Completeness, Strong Cut Elimination.