Fuzzy Modal Logics: what we (don't) know

Félix Bou

IIIA - CSIC
Barcelona (Spain)
fbou@iiia.csic.es

September 11th
LATD 2010

Outline

1. Classical Modal Logic
2. Fuzzy Modal Logic: Introduction
3. Fuzzy Modal Logic: Decidability
4. Fuzzy Modal Logic: Axiomatizations
5. Fuzzy Modal Logic: Idempotent Frames
6. Fuzzy Modal Logic: Final Remarks

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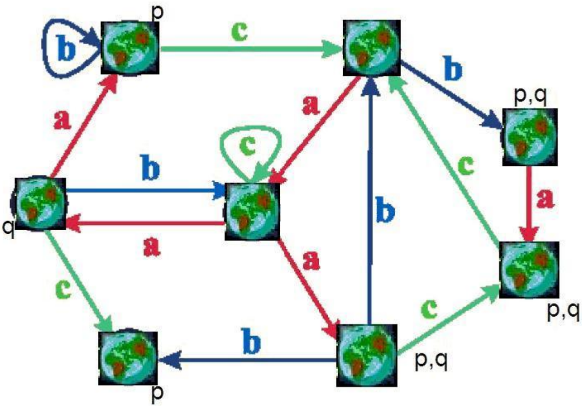

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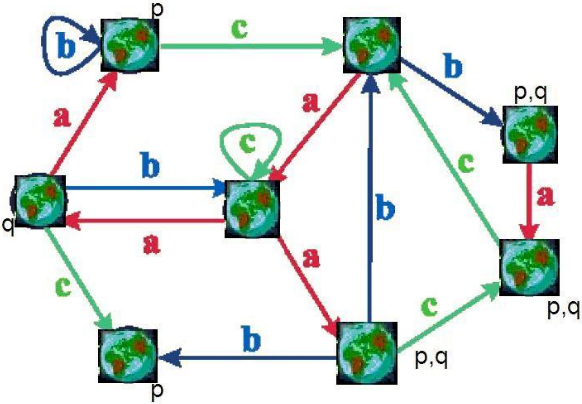

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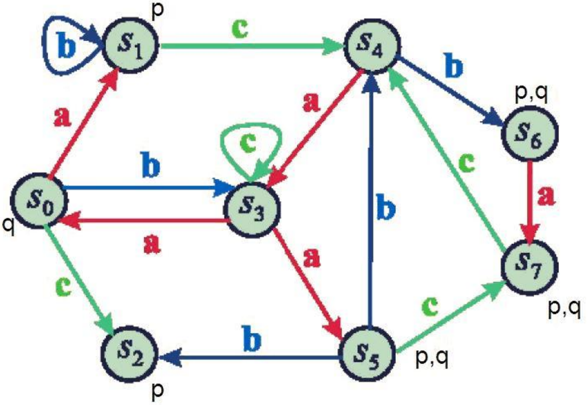

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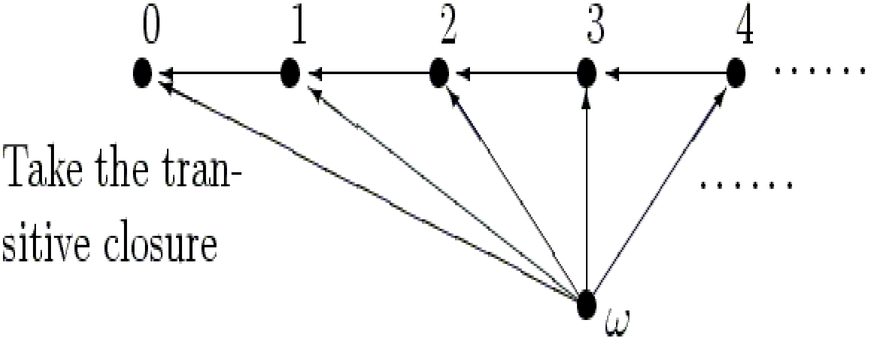

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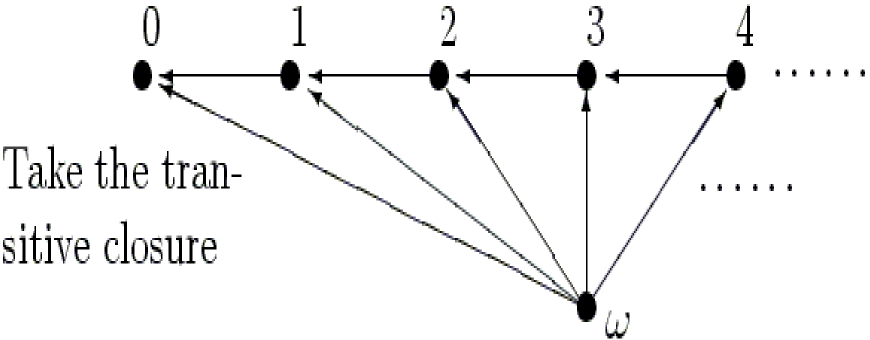

- Harmless Simplicity Assumption: only 1 binary relation R.

- Modal formulas: FOL formulas $\varphi(x)$ built using
  - atomic predicates $(p_1, p_2, , \ldots)$: $P_1 x$, $P_2 x$, ...
  - propositional connectives: $\wedge, \vee, \rightarrow, \neg$.
  - Universal Bounded Quant. ($\Box$):

$$\varphi(x) \quad \triangleright \quad \forall y (Rxy \rightarrow \varphi(y))$$

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Modal Logics (picture from [Chagrov and Zakharyaschev])

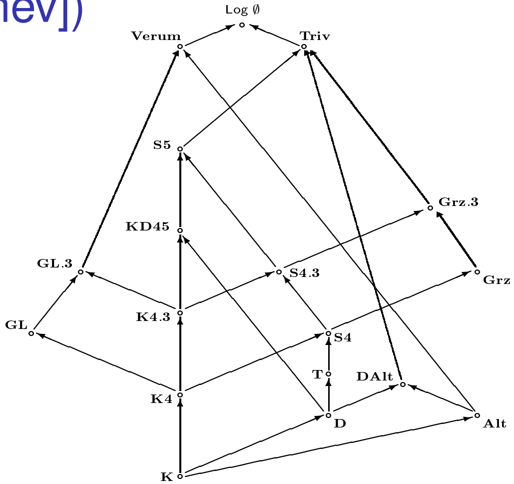

Figure 1.1: Lattice of 'standard' modal logics.

Constructions preserving the value of modal formulas

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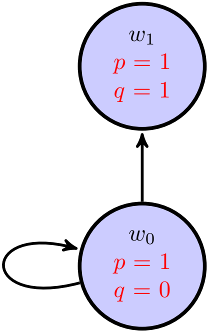

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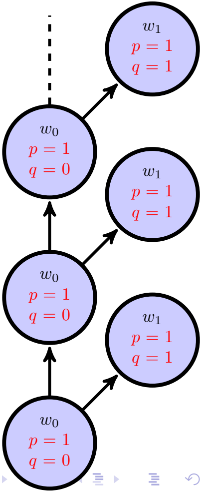

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Axiomatizability

- The non-modal consequence relation $\vdash_{\mathbf{2}}$ is defined by

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- Let us assume that we know how to characterize these maps; in the sense that given a non-modal homomorphism $h : \mathbf{Fm} \longrightarrow \mathbf{2}$ it holds that
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  Thus, the minimum modal logic (i.e., K) is the closure of L under $\vdash_{\mathbf{2}}$.

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- A set L satisfying the previous conditions is the smallest set containing
  - propositional tautologies,
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- $\Gamma \vdash^{g} \varphi \quad$ iff $\quad \{\Box^{n}\gamma : n \in \omega, \gamma \in \Gamma\} \vdash^{l} \varphi$.

Motivation

- The final aim is developing (uni)modal logics over many-valued logics, but as a first step we need to focus on the minimum logic.
- The modal language provides a fragment were it could be the case that the complexity of standard fuzzy logics keeps low (even at the decidable level).

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- The modal language is the expansion of the non-modal one with a unary operators $\Box$ and $\Diamond$.
- (Many-valued) Kripke models are triples $\langle W, R, V \rangle$ where W is a set (of worlds), $R : W \times W \longrightarrow A$ and $V : Fm \times W \longrightarrow A$ such that for every world w,
  1. $V(\bullet, w)$ is a non-modal homomorphism,
  2. $V(\Box\varphi, w) = \bigwedge\{R(w, w') \rightarrow V(\varphi, w') : w' \in W\}$,
  3. $V(\Diamond\varphi, w) = \bigvee\{R(w, w') \odot V(\varphi, w') : w' \in W\}$,

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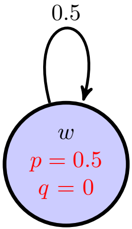

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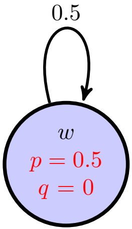

$\Box(p \rightarrow q) = 0.5 \rightarrow (0.5 \rightarrow 0) = 1$ (in $\mathbf{L_3}$)
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- CFr is in general not definable by modal axioms even if there are canonical constants. This is so because indeed $\mathbf{\Lambda}(\mathrm{BFr}, \mathbf{A}^c) = \mathbf{\Lambda}(\mathrm{CFr}, \mathbf{A}^c)$ (when **A** is finite).

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- What is the complexity?

FOL	1Taut	+Taut	1Sat	+Sat
$[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$	Σ_1-c	Σ_1-c	Π_1-c	Π_1-c
$[\mathbf{0},\mathbf{1}]_{\mathbf{L}}$	Π_2-c	Σ_1-c	Π_1-c	Σ_2-c
$[\mathbf{0},\mathbf{1}]_{\mathbf{P}}$	non arith.	non arith.	non arith.	non arith.

Gödel Case (Olivetti-Metcalfe)

$[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$	1Taut	$+$Taut	1Sat	$+$Sat
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- The modal fragment with only $\Box$ (i.e., without $\Diamond$) is decidable, and it is Pspace-c.

Lukasiewicz Case (Hájek)

$[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$	1Taut	$+$Taut	1Sat	$+$Sat
FOL	Π_2-c	Σ_1-c	Π_1-c	Σ_2-c
Modal	dec.	dec.	dec.	dec.

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How to axiomatize these modal logics?

- The non-modal consequence relation $\vdash_{\mathbf{A}}$ is defined by

$$\Gamma \vdash_{\mathbf{A}} \varphi \iff \forall h \in \mathrm{Hom}(\mathbf{F}m, \mathbf{A}), \text{ if } h[\Gamma] \subseteq \{1\} \text{ then } h(\varphi) = 1.$$

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Theorem (Caicedo-Rodríguez, Metcalfe-Olivetti)

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So, $\mathbf{\Lambda}(\mathrm{Fr}, [\mathbf{0}, \mathbf{1}]_{\mathbf{G}})$ coincides with this set L. And also $\mathbf{\Lambda}(\mathrm{CFr}, [\mathbf{0}, \mathbf{1}]_{\mathbf{G}})$ is the same set.

The case of a finite MV chain $\mathbf{L_n}$

Theorem (Hansoul-Teheux)

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So, $\mathbf{\Lambda}(\mathrm{CFr}, \mathbf{L_n})$ coincides with this set L.

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So, $\mathbf{\Lambda}(\mathrm{Fr}, \mathbf{L_3})$ coincides with this set L. It is not enough to add $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ in order to obtain $\mathbf{\Lambda}(\mathrm{CFr}, \mathbf{L_3})$.

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where $a_2 = \frac{1}{n-1}$, $a_3 = \frac{2}{n-1}$, ..., $a_{n-1} = \frac{n-2}{n-1}$ and $a_n = 1$.

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- If moreover we assume that $\mathbf{A^c}$ has a unique coatom k, then adding the normality axiom plus $\Box(\overline{k} \vee \varphi) \to (\overline{k} \vee \Box\varphi)$ it is enough to axiomatize $\mathbf{\Lambda}(\mathrm{CFr}, \mathbf{A^c})$.

Coming back to the Gödel case

- It is known that over the standard Godel algebra, Kripke models are indistinguishable from crisp ones when we only allow $\Box$, i.e., $\mathbf{\Lambda}(\mathrm{IFr}, [\mathbf{0}, \mathbf{1}]_{\mathbf{G}}) = \mathbf{\Lambda}(\mathrm{CFr}, [\mathbf{0}, \mathbf{1}]_{\mathbf{G}})$.

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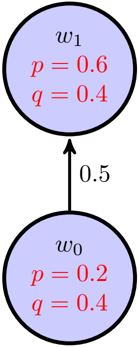

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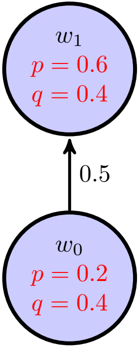

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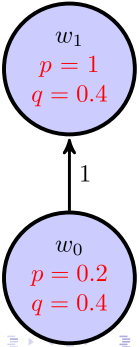

Converting a Kripke model into a crisp one in the Gödel case

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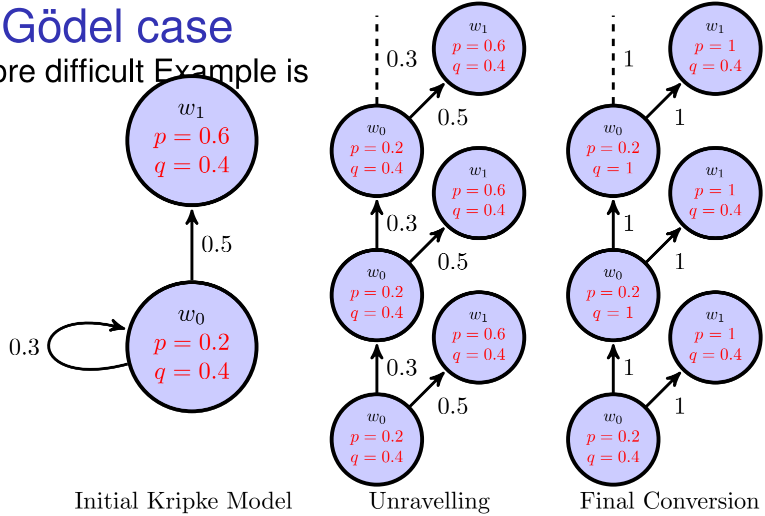

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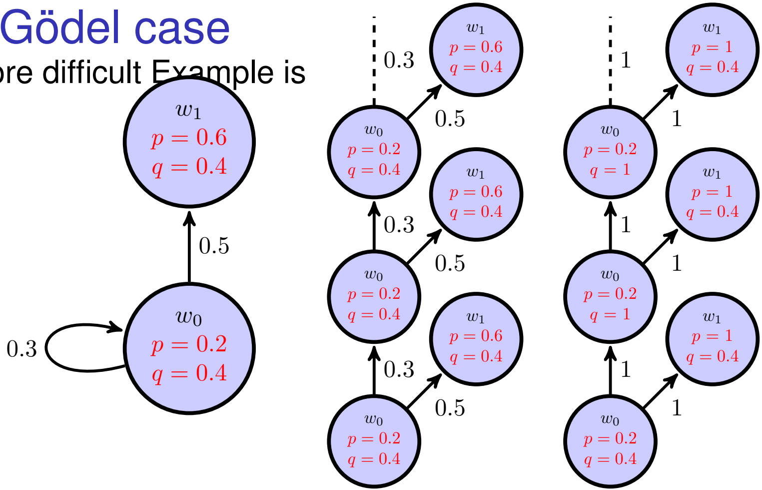

Initial Kripke Model Unravelling Final Conversion

Unfortunately this method does not work. The value of $\Box(q \to \Box q)$ is, at the root, 1 on the left Kripke model and 0.4 on the right one.

A good method to convert a tree into a crisp Kripke model

Definition

Let us assume that $\langle W, R, V\rangle$ is a tree and w_0 is its root. The Kripke model $\mathfrak{M}' = \langle W', R', V'\rangle$ is defined as the one such that

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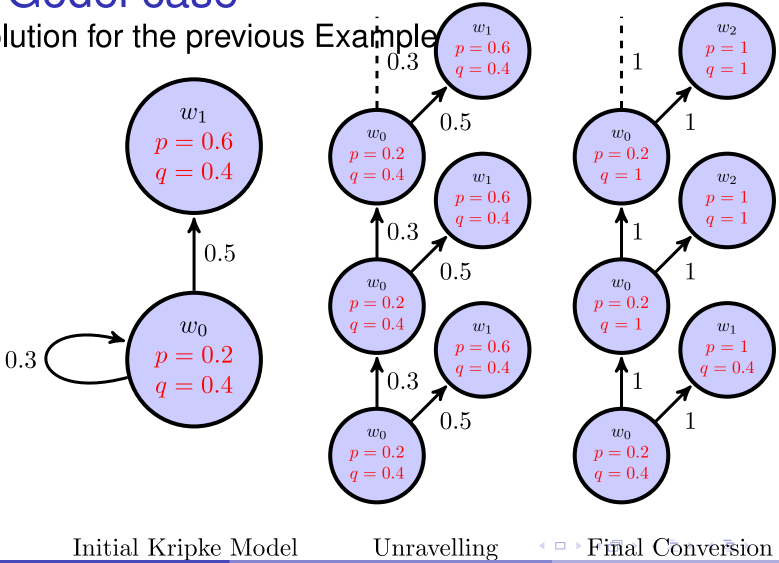
- $V'(w, p) := \begin{cases} 1, & \text{if } \eta(w) \leqslant V(w, p) \\ V(w, p), & \text{otherwise} \end{cases}$

where $w_0, w_1, \ldots, w_n, w$ is the unique path from w_0 to w, and $\eta(w) = R(w_0, w_1) \wedge \ldots \wedge R(w_n, w)$.

Converting a Kripke model into a crisp one in the Gödel case

A solution for the previous Example



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Sketch of the Proof

By induction on the modal depth of modal formulas we prove that for every modal formula φ, it simultaneously holds for every $w \in W$ that

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Then, taking $w = w_0$ (we remind that $\eta(w_0) = 1$) we get that

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Consequences of the previous construction

Theorem (only with $\Box$)

Let $h : \mathbf{Fm} \longrightarrow \mathbf{A}$ be a non-modal homomorphism and $\mathbf{A}$ be a complete BL chain. The following statements are equivalent.

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Informal Interpretation of the Previous Result

On the BL framework if the normality axioms holds then indeed we are only considering crisp Kripke models.

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6. Can we characterize which first-order formulas are equivalent to a modal formula?