Fuzzy Modal Logics: what we (don’t) know

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Outline

1. Classical Modal Logic
2. Fuzzy Modal Logic: Introduction
3. Fuzzy Modal Logic: Decidability
4. Fuzzy Modal Logic: Axiomatizations
5. Fuzzy Modal Logic: Idempotent Frames
6. Fuzzy Modal Logic: Final Remarks
FOL language with

- Unary Predicates:
- Binary Relations:

Structures for this FOL language are very common:

- graphs, trees, automata, abstract models comput., sets, etc.

Harmless Simplicity Assumption: only 1 binary relation $R$. 

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![Diagram with labeled nodes and arrows representing relationships between elements.]
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![Diagram](image-url)
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Take the transitive closure

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Modal formulas: FOL formulas \( \varphi(x) \) built using

- atomic predicates \((p_1, p_2, \ldots)\): \( P_1 x, P_2 x, \ldots \)
- propositional connectives: \( \land, \lor, \rightarrow, \neg \).
- Universal Bounded Quant. \( (\Box) \):
  \[
  \varphi(x) \rightarrow \forall y (R xy \rightarrow \varphi(y))
  \]
- Existential Bounded Quant. \( (\Diamond) \):
  \[
  \varphi(x) \rightarrow \exists y (R xy \land \varphi(y))
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- propositional connectives: $\land, \lor, \rightarrow, \neg$.
- Universal Bounded Quant. ($\square$):
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Examples:
- $\forall y (\neg Rxy \rightarrow Py)$
- $\forall y (Rxy \rightarrow \neg Py)$
- $\forall y (Rxy \rightarrow \exists z Pz)$
- $\forall y (Rxy \rightarrow \exists z (Ryz \land Pz))$
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- $\forall y(Rxy \to \exists z(Ryz \land Pz))$
- $\square \Diamond p$
- $\forall y(Rxy \to (Py \to Qy)) \to (\forall y(Rxy \to Py) \to \forall y(Rxy \to Qy))$
- $\Box (p \to q) \to (\Box p \to \Box q)$
Modal Logics (picture from [Chagrov and Zakharyaschev])

Figure 1.1: Lattice of ‘standard’ modal logics.
Constructions preserving the value of modal formulas

- If we replace a Kripke model (with a distinguished node) by a bisimilar one we do not modify the value of modal formulas. Indeed, bisimulations play an analogous role to the one played by ultraproducts for the first-order language.
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- It is undecidable to determine if a first-order formula with one free variable is equivalent to a modal one.
Unravelling: Trees are enough

- The unravelling is always a tree and it is indistinguishable from the original Kripke model by the modal language.
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\[
\begin{align*}
&w_0 \\
p &= 1 \\
q &= 0 \\
&w_1 \\
p &= 1 \\
q &= 1
\end{align*}
\]
Decidability of the validity of modal formulas

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- Validity is Pspace-c.
Axiomatizability

The non-modal consequence relation $\vdash_2$ is defined by

$$\Gamma \vdash_2 \varphi \iff \forall h \in \text{Hom}(Fm, 2), \text{ if } h[\Gamma] \subseteq \{1\} \text{ then } h(\varphi) = 1.$$
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- Every world $w \in W$ in a Kripke model can be identified with the map $V(\bullet, w) : Fm \rightarrow 2$.
- Let us assume that we know how to characterize these maps; in the sense that given a non-modal homomorphism $h : Fm \rightarrow 2$ it holds that
  - $h$ is semantically arising from a Kripke model, iff
  - $h[L] = 1$ where $L$ is the set of formulas . . .
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- $h$ is semantically arising from a Kripke model, iff
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Thus, the minimum modal logic (i.e., K) is the closure of $L$ under $\vdash_2$. 

Axiomatizability

A set \( L \) satisfying the previous conditions is the smallest set containing

- propositional tautologies,
- \( \Box(\varphi \land \psi) \leftrightarrow (\Box \varphi \land \Box \psi) \) [or \( \Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \)]

and being closed under

- Modus Ponens,
- Generalization Rule: \( \varphi \vdash \Box \varphi \).
Axiomatizability

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- Modus Ponens,
- Generalization Rule: $\varphi \supset \square \varphi$.

- The closure of this $L$ under $\vdash_2$ is the local modal consequence consequence.
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  and being closed under
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  - Generalization Rule: $\varphi \rightarrow \Box \varphi$.

The closure of this $L$ under $\vdash_2$ is the local modal consequence consequence.

$\Gamma \vdash^g \varphi \iff \{\Box^n \gamma : n \in \omega, \gamma \in \Gamma\} \vdash^l \varphi.$
Motivation

- The final aim is developing (uni)modal logics over many-valued logics, but as a first step we need to focus on the minimum logic.
- The modal language provides a fragment were it could be the case that the complexity of standard fuzzy logics keeps low (even at the decidable level).
Replacing 2 with a residuated lattice $\mathbf{A}$

We fix a complete residuated lattice $\mathbf{A} = \langle \land, \lor, \odot, \rightarrow, 1, 0 \rangle$ (in the sense of a FL$_{ew}$ algebra).
Replacing 2 with a residuated lattice $A$

- We fix a complete residuated lattice $A = \langle \land, \lor, \otimes, \rightarrow, 1, 0 \rangle$ (in the sense of a FL$_{ew}$ algebra). Sometimes we will add canonical constants $\bar{a}$, and then we will write $A^c$. 
Replac**ing $\mathbf{2}$ with a residuated lattice $\mathbf{A}$**

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- The **modal language** is the expansion of the non-modal one with a unary operators $\square$ and $\Diamond$. 
Replacing 2 with a residuated lattice $\mathbf{A}$

- We fix a complete residuated lattice $\mathbf{A} = \langle \wedge, \vee, \circ, \rightarrow, 1, 0 \rangle$ (in the sense of a $\text{FL}_{ew}$ algebra). Sometimes we will add canonical constants $\bar{a}$, and then we will write $\mathbf{A}^c$.

- The modal language is the expansion of the non-modal one with a unary operators $\Box$ and $\Diamond$.

- (Many-valued) Kripke models are triples $\langle W, R, V \rangle$ where $W$ is a set (of worlds), $R : W \times W \rightarrow A$ and $V : Fm \times W \rightarrow A$ such that for every world $w$,
  
  1. $V(\bullet, w)$ is a non-modal homomorphism,
  2. $V(\Box \varphi, w) = \wedge \{ R(w, w') \rightarrow V(\varphi, w') : w' \in W \}$,
  3. $V(\Diamond \varphi, w) = \bigvee \{ R(w, w') \circ V(\varphi, w') : w' \in W \}$,
Some Examples of Valid Formulas

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- $\square 1,$
- $\square (p \land q) \leftrightarrow (\square p \land \square q),$ [using $x \to (y \land z) \approx (x \to y) \land (x \to z)$]
Some Examples of Valid Formulas

- $\square 1$,
- $\square (p \land q) \leftrightarrow (\square p \land \square q)$, [using $x \rightarrow (y \land z) \approx (x \rightarrow y) \land (x \rightarrow z)$]
- $\neg \square p \rightarrow \square \neg \neg p$, 
Some Examples of Valid Formulas

- □1,
- □(p ∧ q) ↔ (□p ∧ □q), [using x → (y ∧ z) ≈ (x → y) ∧ (x → z)]
- ¬¬□p → □¬¬p,
- □(a → φ) ↔ (a → □φ),
Some Examples of Valid Formulas

- $\Box 1,$
- $\Box (p \land q) \leftrightarrow (\Box p \land \Box q),$ [using $x \rightarrow (y \land z) \approx (x \rightarrow y) \land (x \rightarrow z)$]
- $\neg \neg \Box p \rightarrow \Box \neg \neg p,$
- $\Box (\overline{a} \rightarrow \varphi) \leftrightarrow (\overline{a} \rightarrow \Box \varphi),$ [using $\bigwedge_i (x \rightarrow y_i) \approx x \rightarrow \bigwedge_i y_i$]
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- $\neg \neg \Box p \rightarrow \Box \neg \neg p$,
- $\Box(\bar{a} \rightarrow \varphi) \iff (\bar{a} \rightarrow \Box \varphi)$, [using $\bigwedge_i (x \rightarrow y_i) \approx x \rightarrow \bigwedge_i y_i$]
- Unfortunately, sometimes the normality axiom $\Box (p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ is not valid
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  \( (\Box p \odot \Box q) \rightarrow \Box (p \odot q) \)).

\[\begin{array}{c}
w \\
p = 0.5 \\
q = 0
\end{array}\]
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- □1,
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- Unfortunately, sometimes the normality axiom □(p → q) → (□p → □q) is not valid (the same for (□p ⊙ □q) → □(p ⊙ q)).

\[
\begin{align*}
\text{w} & \quad p = 0.5 \\
& \quad q = 0 \\
\end{align*}
\]

\[\begin{align*}
\square(p \rightarrow q) &= 0.5 \rightarrow (0.5 \rightarrow 0) = 1 \text{ (in } L_3) \\
\square p &= 0.5 \rightarrow 0.5 = 1 \\
\square q &= 0.5 \rightarrow 0 = 0.5
\end{align*}\]
Some classes of frames over $A$

- $\mathbf{Fr}$: the class of all Kripke frames.
Some classes of frames over $A$

- **Fr**: the class of all Kripke frames.
- **IFr**: the class of idempotent Kripke frames (i.e., $R(w, w') = R(w, w') \circ R(w, w')$).
- **BFr**: the class of Boolean Kripke frames (i.e., $R(w, w') \lor \neg R(w, w') = 1$ and $R(w, w') \land \neg R(w, w') = 0$).
- **CFr**: the class of crisp Kripke frames (i.e., $R(w, w') \in \{0, 1\}$).

To denote the set of valid formulas in these classes of frames we use the notation $\Lambda(Fr, A)$, $\Lambda(IFr, A)$, $\Lambda(IFr, A_c)$, ...
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To denote the set of valid formulas in these classes of frames we use the notation $\Lambda(Fr, A)$, $\Lambda(Fr, A^c)$, $\Lambda(IFr, A)$, $\Lambda(IFr, A^c)$, ...
Modal Characterization (Esteva-Godo-Rodríguez-B.)

- IFr is definable by the normality axiom
  \( \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) \).

- CFr and BFr are in general not definable by modal axioms.
  BFr (when \( A \) is finite) is definable by modal axioms if we allow canonical constants.
  Then, it is definable by the set \( \{ 2(k \lor p) \rightarrow (k \lor 2p) : k \in \text{CoAtom}(A) \} \).

- CFr is in general not definable by modal axioms even if there are canonical constants.

This is so because indeed \( \Lambda(BFr, A) = \Lambda(CFr, A) \) (when \( A \) is finite).
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Modal Characterization (Esteva-Godo-Rodríguez-B.)

- IFr is definable by the normality axiom\( \Box (p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) \). And also by \( (\Box p \odot \Box q) \rightarrow \Box (p \odot q) \) and \( (\Box p \odot \Box p) \rightarrow \Box (p \odot p) \).
- CFr and BFr are in general not definable by modal axioms.
- BFr (when \( A \) is finite) is definable by modal axioms if we allow canonical constants. Then, it is definable by the set \( \{ \Box (\bar{k} \lor p) \rightarrow (\bar{k} \lor \Box p) : k \in CoAtom(A) \} \).
- CFr is in general not definable by modal axioms even if there are canonical constants. This is so because indeed \( \Lambda(BFr, A^c) = \Lambda(CFr, A^c) \) (when \( A \) is finite).
Decidability in the finite case

If $A$ finite,

- $\Lambda(Fr, A^c)$ is decidable (filtration method, trees, etc).
Decidability in the finite case

If $A$ finite,

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Decidability in the finite case

If \( A \) finite,

- \( \Lambda(Fr, A^c) \) is decidable (filtration method, trees, etc).
- Filtration gives a trivial NExp algorithm for validity.
- What is the complexity?
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<tr>
<th>FOL</th>
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- Finite Model Property fails ($\neg \Box p \land \neg \Diamond \neg p$).
- The modal fragment with only $\Box$ (i.e., without $\Diamond$) is decidable, and it is Pspace-c.
## Lukasiewicz Case (Hájek)

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Finite Model Property holds (using witnessed models).
Lukasiewicz Case (Hájek)

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  - $\varphi \in +\text{Taut}$ iff $T(\varphi) \vDash \varphi$,  
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What is the complexity?

Félix Bou (IIIA - CSIC)
Lukasiewicz Case (Hájek)

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- What is the complexity?
## Product Case (Cerami-Esteva-B.)

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Finite Model Property fails ($\neg\Box p \land \neg\Diamond\neg p$, $\Box p \land \neg\Box(p \odot p)$).

### Product Case (Cerami-Esteva-B.)

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- Finite Model Property fails ($\neg \square p \land \neg \Diamond \neg p$, $\square p \land \neg \square (p \odot p)$).
- $\varphi \in +\text{Sat}$ iff $\neg \varphi \notin 1\text{Taut}$.
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How to axiomatize these modal logics?

The non-modal consequence relation $\vdash_A$ is defined by

$$\Gamma \vdash_A \varphi \iff \forall h \in \text{Hom}(Fm, A), \text{ if } h[\Gamma] \subseteq \{1\} \text{ then } h(\varphi) = 1.$$
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- It is easy to prove that for every $K \in \{\text{Fr}, \text{IFr}, \text{CFr}\}$ there is some set $L$ such that

1. $h : \mathsf{Fm} \rightarrow A$ is arising from a Kripke model in $K$, iff
2. $h : \mathsf{Fm} \rightarrow A$ is a non-modal homomorphism such that $h[L] = \{1\}$. 

Hence, $\Lambda(K, A)$ is the closure of $L$ under $\vdash_A$. The same holds when there are canonical constants.
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- Analogously, $\vdash_{A^c}$ is defined by

$$\Gamma \vdash_{A^c} \varphi \iff \forall h \in \text{Hom}(Fm, A^c), \text{ if } h[\Gamma] \subseteq \{1\} \text{ then } h(\varphi) = 1.$$ 

- It is easy to prove that for every $K \in \{\text{Fr}, \text{IFr}, \text{CFr}\}$ there is some set $L$ such that

1. $h : Fm \rightarrow A$ is arising from a Kripke model in $K$, iff
2. $h : Fm \rightarrow A$ is a non-modal homomorphism such that $h[L] = \{1\}$.

Hence, $\Lambda(K, A)$ is the closure of $L$ under $\vdash_{A}$.

- The same holds when there are canonical constants.
Standard Gödel Algebra \([0, 1]_G\) (only with \(\square\))

Theorem (Caicedo-Rodríguez, Metcalfe-Olivetti)

Let \(h : Fm \longrightarrow [0, 1]_G\) be a non-modal homomorphism. The following statements are equivalent.

1. \(h\) is semantically arising from a Kripke model,
Standard Gödel Algebra $[\mathbf{0}, \mathbf{1}]_G$ (only with $\Box$)

**Theorem (Caicedo-Rodríguez, Metcalfe-Olivetti)**

Let $h : \text{Fm} \longrightarrow [\mathbf{0}, \mathbf{1}]_G$ be a non-modal homomorphism. The following statements are equivalent.

1. $h$ is semantically arising from a Kripke model,
2. $h[L] = 1$ where $L$ is the smallest set such that it
   (i) is closed under $\vdash [\mathbf{0}, \mathbf{1}]_G$,
   (ii) contains the axioms $\Box \mathbf{1}$, $\Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$ and
   $\neg \neg \Box \varphi \rightarrow \Box \neg \neg \varphi$,
   (iii) is closed under the Necessity rule $\frac{\varphi}{\Box \varphi}$ (N)

$$\Rightarrow$$

$h$ is semantically arising from a crisp Kripke model.

So, $\Lambda(\text{Fr}, [\mathbf{0}, \mathbf{1}]_G)$ coincides with this set $L$.

And also $\Lambda(\text{CFr}, [\mathbf{0}, \mathbf{1}]_G)$ is the same set.
Standard Gödel Algebra $[0, 1]_G$ (only with $\Box$)

Theorem (Caicedo-Rodríguez, Metcalfe-Olivetti)

Let $h : \text{Fm} \longrightarrow [0, 1]_G$ be a non-modal homomorphism. The following statements are equivalent.

1. $h$ is semantically arising from a Kripke model,
2. $h[L] = 1$ where $L$ is the smallest set such that it
   (i) is closed under $\vdash [0,1]_G$,
   (ii) contains the axioms $\Box 1$, $\Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$ and
        $\neg \neg \Box \varphi \rightarrow \Box \neg \neg \varphi$,
   (iii) is closed under the Necessity rule $\frac{\varphi}{\Box \varphi}$ (N)
3. $h$ is semantically arising from a crisp Kripke model.
Theorem (Caicedo-Rodríguez, Metcalfe-Olivetti)

Let \( h : Fm \rightarrow [0, 1]_G \) be a non-modal homomorphism. The following statements are equivalent.

1. \( h \) is semantically arising from a Kripke model,
2. \( h[L] = 1 \) where \( L \) is the smallest set such that it
   (i) is closed under \( \vdash_{[0,1]_G} \),
   (ii) contains the axioms \( \Box 1 \), \( \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \) and \( \neg\neg \Box \varphi \rightarrow \Box \neg\neg \varphi \),
   (iii) is closed under the Necessity rule \( \frac{\varphi}{\Box \varphi} \) (N)
3. \( h \) is semantically arising from a crisp Kripke model.

So, \( \Lambda(Fr, [0, 1]_G) \) coincides with this set \( L \).
Standard Gödel Algebra $[0,1]_G$ (only with $\Box$)

**Theorem (Caicedo-Rodríguez, Metcalfe-Olivetti)**

Let $h : \text{Fm} \longrightarrow [0,1]_G$ be a non-modal homomorphism. The following statements are equivalent.

1. $h$ is semantically arising from a Kripke model,
2. $h[L] = 1$ where $L$ is the smallest set such that it
   (i) is closed under $\vdash_{[0,1]_G}$,
   (ii) contains the axioms $\Box 1$, $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$ and $\neg \neg \Box \varphi \rightarrow \Box \neg \neg \varphi$,
   (iii) is closed under the Necessity rule $\varphi \quad \Box \varphi$ (N)
3. $h$ is semantically arising from a crisp Kripke model.

So, $\Lambda(\text{Fr}, [0,1]_G)$ coincides with this set $L$. And also $\Lambda(\text{CFr}, [0,1]_G)$ is the same set.
The case of a finite MV chain $L_n$

Theorem (Hansoul-Teheux)

Let $h : \mathbf{Fm} \longrightarrow L_n$ be a non-modal homomorphism. The following statements are equivalent.

1. $h$ is semantically arising from a crisp Kripke model,

2. $L$ is the smallest set such that it

(i) is closed under $\vdash L_n$,

(ii) contains the axioms $2_1$, $2_2 (\phi \rightarrow \psi) \rightarrow (2_1 \phi \rightarrow 2_1 \psi)$,

2. $(\phi \oplus \phi) \leftrightarrow (2_1 \phi \oplus 2_1 \phi)$ and $2_3 (\phi \otimes \phi) \leftrightarrow (2_1 \phi \otimes 2_1 \phi)$,

(iii) is closed under the Necessity rule $\phi (N)$ $2_1 \phi$

So, $\Lambda(\text{CFr}, L_n)$ coincides with this set $L$. 
The case of a finite MV chain $L_n$

**Theorem (Hansoul-Teheux)**

Let $h : \text{Fm} \rightarrow L_n$ be a non-modal homomorphism. The following statements are equivalent.

1. $h$ is semantically arising from a crisp Kripke model,
2. $h[L] = 1$ where $L$ is the smallest set such that it
   
   (i) is closed under $\vdash_{L_n}$,
   
   (ii) contains the axioms $\Box 1$, $\Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$,
   
   $\Box (\varphi \oplus \varphi) \leftrightarrow (\Box \varphi \oplus \Box \varphi)$ and $\Box (\varphi \odot \varphi) \leftrightarrow (\Box \varphi \odot \Box \varphi)$,

   (iii) is closed under the Necessity rule $\varphi \rightarrow \Box \varphi$ (N)
The case of a finite MV chain $L_n$

**Theorem (Hansoul-Teheux)**

Let $h : \mathbf{Fm} \rightarrow L_n$ be a non-modal homomorphism. The following statements are equivalent.

1. $h$ is semantically arising from a *crisp* Kripke model,
2. $h[L] = 1$ where $L$ is the smallest set such that it
   - is closed under $\vdash_{L_n}$,
   - contains the axioms $\Box 1$, $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$,
   - $\Box(\varphi \oplus \varphi) \leftrightarrow (\Box \varphi \oplus \Box \varphi)$ and $\Box(\varphi \odot \varphi) \leftrightarrow (\Box \varphi \odot \Box \varphi)$,
   - is closed under the Necessity rule $\frac{\varphi}{\Box \varphi}$ (N)

So, $\Lambda(CFr, L_n)$ coincides with this set $L$. 
The case of standard MV algebra $[0, 1]_L$

Theorem (Hansoul-Teheux)

Let $h : \text{Fm} \rightarrow [0, 1]_L$ be a non-modal homomorphism. The following statements are equivalent.

1. $h$ is semantically arising from a crisp Kripke model,

2. $h[0, 1]_L = 1$ where $L$ is the smallest set such that it
   (i) is closed under the properties in the previous slide (of course now using $\vdash [0, 1]_L$),
   (ii) contains the axioms $2(\phi \oplus \phi \leq n)$ for every $n \in \omega$,
   (iii) is closed under the infinitary rule $\phi \oplus \phi, \phi \oplus \phi, ..., \phi \oplus \phi, ...$ ($\text{InfGreat}$).

So, $\Lambda(\text{CFr}, [0, 1]_L)$ coincides with this set $L$. 
The case of standard MV algebra $[0, 1]_L$

Theorem (Hansoul-Teheux)

Let $h : \text{Fm} \rightarrow [0, 1]_L$ be a non-modal homomorphism. The following statements are equivalent.

1. $h$ is semantically arising from a crisp Kripke model,
2. $h[L] = 1$ where $L$ is the smallest set such that it is closed under the properties in the previous slide (of course now using $\lceil_{[0,1]_L}$),
   (i) is closed under the properties in the previous slide (of course now using $\lceil_{[0,1]_L}$),
   (ii) contains the axioms $\Box(\varphi \oplus \varphi^n) \leftrightarrow ((\Box \varphi) \oplus (\Box \varphi)^n)$ (for every $n \in \omega$),
   (iii) is closed under the infinitary rule

$$\varphi \oplus \varphi, \varphi \oplus \varphi^2, \ldots, \varphi \oplus \varphi^n, \ldots$$

($\text{InfGreat}$)
The case of standard MV algebra $[0, 1]_L$

**Theorem (Hansoul-Teheux)**

Let $h : \mathbf{Fm} \rightarrow [0, 1]_L$ be a non-modal homomorphism. The following statements are equivalent.

1. $h$ is semantically arising from a crisp Kripke model,
2. $h[L] = 1$ where $L$ is the smallest set such that it
   - is closed under the properties in the previous slide (of course now using $\vdash [0, 1]_L$),
   - contains the axioms $\Box(\varphi \oplus \varphi^n) \leftrightarrow ((\Box \varphi) \oplus (\Box \varphi)^n)$ (for every $n \in \omega$),
   - is closed under the infinitary rule
     \[ \varphi \oplus \varphi, \varphi \oplus \varphi^2, \ldots, \varphi \oplus \varphi^n, \ldots \]
     (InfGreat)

So, $\Lambda(\mathbf{CFr}, [0, 1]_L)$ coincides with this set $L$. 
The case of \( L_3 \) (Esteva-Godo-Rodríguez-B.)

Let \( h : \text{Fm} \longrightarrow L_3 \) be a non-modal homomorphism. Then,

1. \( h \) is semantically arising from a Kripke model, iff
The case of $L_3$ (Esteva-Godo-Rodríguez-B.)

Let $h : \text{Fm} \longrightarrow L_3$ be a non-modal homomorphism. Then,

1. $h$ is semantically arising from a Kripke model, iff
2. $h[L] = 1$ where $L$ is the smallest set such that it
   (i) is closed under $\vdash_{L_3}$, and contains the axioms $\Box 1$ and $\Box(\varphi \land \psi) \leftrightarrow (\Box \varphi \land \Box \psi)$,
   (ii) is closed under the Monotonicity rule $\frac{\varphi \rightarrow \psi}{\Box \varphi \rightarrow \Box \psi}$ (Mon),
   (iii) is closed under the rules (where $\eta_{0.5}(\varphi) := \varphi \oplus \varphi$ and $\eta_1(\varphi) := \varphi \odot \varphi$)

\[ \begin{align*}
(\eta_{0.5}(\varphi_2) \land \eta_1(\varphi_3)) & \rightarrow \eta_{0.5}(\varphi) \quad \text{(R0.5)} \\
(\eta_{0.5}(\Box \varphi_2) \land \eta_1(\Box \varphi_3)) & \rightarrow \eta_{0.5}(\Box \varphi) \\
(\eta_0(\varphi_2) \land \eta_{0.5}(\varphi_3)) & \rightarrow \eta_{0.5}(\varphi) \\
(\eta_{0.5}(\Box \varphi_2) \land \eta_1(\Box \varphi_3)) & \rightarrow \eta_1(\Box \varphi) \quad \text{(R1)}
\end{align*} \]
The case of $\mathbb{L}_3$ (Esteva-Godo-Rodríguez-B.)

Let $h : \text{Fm} \rightarrow \mathbb{L}_3$ be a non-modal homomorphism. Then,

1. $h$ is semantically arising from a Kripke model, iff
2. $h[L] = 1$ where $L$ is the smallest set such that it
   (i) is closed under $\vdash_{\mathbb{L}_3}$, and contains the axioms $\Box 1$ and $\Box (\varphi \land \psi) \leftrightarrow (\Box \varphi \land \Box \psi)$,
   (ii) is closed under the Monotonicity rule $\frac{\varphi \rightarrow \psi}{\Box \varphi \rightarrow \Box \psi}$ (Mon),
   (iii) is closed under the rules (where $\eta_{0.5}(\varphi) := \varphi \oplus \varphi$ and $\eta_1(\varphi) := \varphi \odot \varphi$)

\[
\frac{(\eta_{0.5}(\varphi_2) \land \eta_1(\varphi_3)) \rightarrow \eta_{0.5}(\varphi)}{(\eta_{0.5}(\Box \varphi_2) \land \eta_1(\Box \varphi_3)) \rightarrow \eta_{0.5}(\Box \varphi)} \quad (R_{0.5})
\]
\[
\frac{(\eta_0(\varphi_2) \land \eta_{0.5}(\varphi_3)) \rightarrow \eta_{0.5}(\varphi) \quad (\eta_{0.5}(\varphi_2) \land \eta_1(\varphi_3)) \rightarrow \eta_1(\varphi)}{(\eta_{0.5}(\Box \varphi_2) \land \eta_1(\Box \varphi_3)) \rightarrow \eta_1(\Box \varphi)} \quad (R_1)
\]

So, $\Lambda(\text{Fr}, \mathbb{L}_3)$ coincides with this set $L$. 
The case of $L_3$ (Esteva-Godo-Rodríguez-B.)

Let $h : \text{Fm} \rightarrow L_3$ be a non-modal homomorphism. Then,

1. $h$ is semantically arising from a Kripke model, iff
2. $h[L] = 1$ where $L$ is the smallest set such that it
   (i) is closed under $\vdash_{L_3}$, and contains the axioms $\square 1$ and $\square(\varphi \land \psi) \leftrightarrow (\square \varphi \land \square \psi)$,
   (ii) is closed under the Monotonicity rule $\frac{\varphi \rightarrow \psi}{\square \varphi \rightarrow \square \psi}$ (Mon),
   (iii) is closed under the rules (where $\eta_{0.5}(\varphi) := \varphi \oplus \varphi$ and $\eta_1(\varphi) := \varphi \odot \varphi$)

$$
\frac{(\eta_{0.5}(\varphi_2) \land \eta_1(\varphi_3)) \rightarrow \eta_{0.5}(\varphi)}{(\eta_{0.5}(\square \varphi_2) \land \eta_1(\square \varphi_3)) \rightarrow \eta_{0.5}(\square \varphi)} \quad (R_{0.5})
$$

$$
\frac{(\eta_0(\varphi_2) \land \eta_{0.5}(\varphi_3)) \rightarrow \eta_{0.5}(\varphi)}{(\eta_0(\square \varphi_2) \land \eta_{0.5}(\square \varphi_3)) \rightarrow \eta_{0.5}(\square \varphi)} \quad (R_1)
$$

So, $\Lambda(\text{Fr}, L_3)$ coincides with this set $L$. It is not enough to add
$\square(\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi)$ in order to obtain $\Lambda(\text{CFr}, L_3)$.
The case of $L_n$ (Esteva-Godo-Rodríguez-B.)

Let $h : Fm \rightarrow L_n$ be a non-modal homomorphism. Then,

1. $h$ is semantically arising from a Kripke model, iff
The case of $L_n$ (Esteva-Godo-Rodríguez-B.)

Let $h : \text{Fm} \rightarrow L_n$ be a non-modal homomorphism. Then,

1. $h$ is semantically arising from a Kripke model, iff
2. $h[L] = 1$ where $L$ is the smallest set such that it
   (i) is closed under $\vdash_{L_n}$, and contains the axioms $\Box 1$ and
   $\Box (\varphi \land \psi) \leftrightarrow (\Box \varphi \land \Box \psi)$,
   (ii) is closed under the Monotonicity rule $\frac{\varphi \rightarrow \psi}{\Box \varphi \rightarrow \Box \psi}$ (Mon),
   (iii) is closed under the rules $(R_a)$ (for every $a \in L_n \setminus \{0\}$)
   \[
   (\eta_{a_2} \odot b(\varphi_2) \land \eta_{a_3} \odot b(\varphi_3) \land \ldots \land \eta_{a_n} \odot b(\varphi_n)) \rightarrow \eta_a \odot b(\varphi) \text{ for all } b > \neg a
   \]
   \[
   (\eta_{a_2}(\Box \varphi_2) \land \eta_{a_3}(\Box \varphi_3) \ldots \land \eta_{a_n}(\Box \varphi_n)) \rightarrow \eta_a(\Box \varphi)
   \]
   where $a_2 = \frac{1}{n-1}$, $a_3 = \frac{2}{n-1}$, ..., $a_{n-1} = \frac{n-2}{n-1}$ and $a_n = 1$. 

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Fuzzy Modal Logics
September 11th, LATD 2010
The case of \( L_n \) (Esteva-Godo-Rodríguez-B.)

Let \( h : \mathbf{Fm} \rightarrow L_n \) be a non-modal homomorphism. Then,

\begin{enumerate}
\item \( h \) is semantically arising from a Kripke model, iff
\item \( h[L] = 1 \) where \( L \) is the smallest set such that it
\hspace{1em} (i) is closed under \( \vdash_{L_n} \), and contains the axioms \( \Box 1 \) and \( \Box (\varphi \land \psi) \leftrightarrow (\Box \varphi \land \Box \psi) \),
\hspace{1em} (ii) is closed under the Monotonicity rule \( \frac{\varphi \rightarrow \psi}{\Box \varphi \rightarrow \Box \psi} \) (Mon),
\hspace{1em} (iii) is closed under the rules \( (R_a) \) (for every \( a \in L_n \setminus \{0\} \))
\hspace{1em} \( (\eta a_2 \odot b(\varphi_2) \land \eta a_3 \odot b(\varphi_3) \land \ldots \land \eta a_n \odot b(\varphi_n)) \rightarrow \eta a \odot b(\varphi) \) for all \( b > -a \)
\hspace{1em} \( (\eta a_2 (\Box \varphi_2) \land \eta a_3 (\Box \varphi_3) \ldots \land \eta a_n (\Box \varphi_n)) \rightarrow \eta a (\Box \varphi) \) (\( R_a \))
\end{enumerate}

where \( a_2 = \frac{1}{n-1} \), \( a_3 = \frac{2}{n-1} \), \ldots, \( a_{n-1} = \frac{n-2}{n-1} \) and \( a_n = 1 \).

So, \( \Lambda(\text{Fr}, L_n) \) coincides with this set \( L \).
The case of $A^c$ with $A$ finite (only with $\Box$)

Let $h : Fm \rightarrow A^c$ be a non-modal homomorphism. Then,

1. $h$ is semantically arising from a Kripke model, iff

   $L$ is the smallest set such that it
   (i) is closed under $\vdash A^c$,
   (ii) contains the axioms
   $(\phi \land \psi) \leftrightarrow (\Box \phi \land \Box \psi)$
   and
   $(\rho \rightarrow \phi) \leftrightarrow (\rho \rightarrow \Box \phi)$
   (iii) is closed under the Monotonicity rule $\phi \rightarrow \psi (\text{Mon})$,

So, $\Lambda(Fr, A^c)$ coincides with this set $L$. 
The case of $A^c$ with $A$ finite (only with $\Box$)

Let $h : \text{Fm} \longrightarrow A^c$ be a non-modal homomorphism. Then,

1. $h$ is semantically arising from a Kripke model, iff
2. $h[L] = 1$ where $L$ is the smallest set such that it
   (i) is closed under $\vdash_{A^c}$,
   (ii) contains the axioms $\Box 1$, $\Box(\varphi \land \psi) \leftrightarrow (\Box \varphi \land \Box \psi)$ and
       $\Box(\overline{a} \rightarrow \varphi) \leftrightarrow (\overline{a} \rightarrow \Box \varphi)$
   (iii) is closed under the Monotonicity rule $\frac{\varphi \rightarrow \psi}{\Box \varphi \rightarrow \Box \psi}$ (Mon),
The case of $A^c$ with $A$ finite (only with $\Box$)

Let $h : \text{Fm} \rightarrow A^c$ be a non-modal homomorphism. Then,

1. $h$ is semantically arising from a Kripke model, iff
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   (i) is closed under $\vdash_{A^c}$,
   (ii) contains the axioms $\Box 1$, $\Box(\varphi \land \psi) \leftrightarrow (\Box \varphi \land \Box \psi)$ and $\Box(\overline{a} \rightarrow \varphi) \leftrightarrow (\overline{a} \rightarrow \Box \varphi)$
   (iii) is closed under the Monotonicity rule $\frac{\varphi \rightarrow \psi}{\Box \varphi \rightarrow \Box \psi}$ (Mon),

So, $\Lambda(\text{Fr}, A^c)$ coincides with this set $L$. 
The case of $\mathcal{A}^c$ with $\mathcal{A}$ finite (only with $\Box$)

- Adding the normality axiom $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$ it is enough to axiomatize $\Lambda(\mathcal{IFr}, \mathcal{A}^c)$.
The case of $\mathcal{A}^c$ with $\mathcal{A}$ finite (only with $\Box$)

- Adding the normality axiom $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$ it is enough to axiomatize $\Lambda(\text{IFr}, \mathcal{A}^c)$.

- If moreover we assume that $\mathcal{A}^c$ has a unique coatom $k$, then adding the normality axiom plus $\Box(k \lor \varphi) \rightarrow (k \lor \Box \varphi)$ it is enough to axiomatize $\Lambda(\text{CFr}, \mathcal{A}^c)$. 
Coming back to the Gödel case

- It is known that over the standard Godel algebra, Kripke models are indistinguishable from crisp ones when we only allow $\Box$, i.e., $\Lambda(\text{IFr}, [0, 1]_G) = \Lambda(\text{CFr}, [0, 1]_G)$. 

How can we transform a Kripke model into a crisp one without modifying the value of modal formulas? An easy example is $w_0 p = 0.2, q = 0.4$.

$w_1 p = 0.6, q = 0.4$ is indistinguishable from $w_0 p = 0.2, q = 0.4$.

$w_1 p = 1, q = 0.4$. 

$w_0 p = 0.2, q = 0.4$.
Coming back to the Gödel case

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- How can we transform a Kripke model into a crisp one without modifying the value of modal formulas?
Coming back to the Gödel case

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- How can we transform a Kripke model into a crisp one without modifying the value of modal formulas? An easy example is

\[
\begin{align*}
\text{w}_0 & \quad p = 0.2 \quad q = 0.4 \\
\text{w}_1 & \quad p = 0.6 \quad q = 0.4
\end{align*}
\]

is indistinguishable from

\[
\begin{align*}
\text{w}_0 & \quad p = 0.2 \quad q = 0.4 \\
\text{w}_1 & \quad p = 1 \quad q = 0.4
\end{align*}
\]
Coming back to the Gödel case

- It is known that over the standard Gödel algebra, Kripke models are indistinguishable from crisp ones when we only allow $\square$, i.e., $\Lambda(\text{IFr}, [0, 1]_G) = \Lambda(\text{CFr}, [0, 1]_G)$.

- How can we transform a Kripke model into a crisp one without modifying the value of modal formulas? An easy example is

$\begin{align*}
\node[draw,fill=blue!20] (w0) at (0,0) {w_0} \\
\node[draw,fill=blue!20] (w1) at (2,0) {w_1} \\
\node (p0) at (w0) {p = 0.2} \\
\node (q0) at (w0) {q = 0.4} \\
\node (p1) at (w1) {p = 0.6} \\
\node (q1) at (w1) {q = 0.4} \\
\node (05) at (0,-1) {0.5}
\end{align*}$

is indistinguishable from

$\begin{align*}
\node[draw,fill=blue!20] (w0) at (0,0) {w_0} \\
\node[draw,fill=blue!20] (w1) at (2,0) {w_1} \\
\node (p0) at (w0) {p = 0.2} \\
\node (q0) at (w0) {q = 0.4} \\
\node (p1) at (w1) {p = 1} \\
\node (q1) at (w1) {q = 0.4} \\
\node (1) at (2,-1) {1}
\end{align*}$
Converting a Kripke model into a crisp one in the Gödel case

A more difficult Example is

\[
\begin{align*}
\text{Initial Kripke Model} & : & w_0 & \rightarrow p = 0.2 & \quad q = 0.4 \\
& & & \downarrow 0.5 \\
& & w_1 & \rightarrow p = 0.6 & \quad q = 0.4 \\
\end{align*}
\]

\[
\begin{align*}
\text{Unravelling} & : & w_0 & \rightarrow p = 0.2 & \quad q = 0.4 \\
& & & \downarrow 0.5 \\
& & w_1 & \rightarrow p = 0.6 & \quad q = 0.4 \\
& & & \downarrow 1 \\
\end{align*}
\]

\[
\begin{align*}
\text{Final Conversion} & : & w_0 & \rightarrow p = 0.2 & \quad q = 0.4 \\
& & & \downarrow 1 \\
& & w_1 & \rightarrow p = 1 & \quad q = 0.4 \\
& & & \downarrow 1 \\
\end{align*}
\]

Unfortunately this method does not work. The value of

\[
(\exists a)(b \rightarrow a)
\]

is, at the root, 1 on the left Kripke model and 0.4 on the right one.

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Converting a Kripke model into a crisp one in the Gödel case

A more difficult Example is

\[ \begin{align*}
  & w_0 \\
  & p = 0.2 \\
  & q = 0.4
\end{align*} \]

\[ \begin{align*}
  & w_1 \\
  & p = 0.6 \\
  & q = 0.4
\end{align*} \]

\[ \begin{align*}
  & 0.3 \\
  & 0.5
\end{align*} \]

Initial Kripke Model

Unravelling

Final Conversion

Unfortunately this method does not work. The value of \( \square (q \rightarrow \square q) \) is, at the root, 1 on the left Kripke model and 0.4 on the right one.
A good method to convert a tree into a crisp Kripke model

Definition

Let us assume that \( \langle W, R, V \rangle \) is a tree and \( w_0 \) is its root. The Kripke model \( \mathcal{M}' = \langle W', R', V' \rangle \) is defined as the one such that

- \( W' \) is the same set \( W \),
- \( R' (w_1, w_2) := \begin{cases} 1, & \text{if } R (w_1, w_2) \neq 0 \\ 0, & \text{if } R (w_1, w_2) = 0 \end{cases} \)
- \( V' (w, p) := \begin{cases} 1, & \text{if } \eta (w) \leq V (w, p) \\ V (w, p), & \text{otherwise} \end{cases} \)

where \( w_0, w_1, \ldots, w_n, w \) is the unique path from \( w_0 \) to \( w \).
A good method to convert a tree into a crisp Kripke model

Definition

Let us assume that $\langle W, R, V \rangle$ is a tree and $w_0$ is its root. The Kripke model $M' = \langle W', R', V' \rangle$ is defined as the one such that

- $W'$ is the same set $W$,
- $R'(w_1, w_2) := \begin{cases} 1, & \text{if } R(w_1, w_2) \neq 0 \\ 0, & \text{if } R(w_1, w_2) = 0 \end{cases}$

where $w_0, w_1, \ldots, w_n$ is the unique path from $w_0$ to $w_n$. and

$\eta(w_0) = R(w_0, w_1) \land \ldots \land R(w_n, w)$.
A good method to convert a tree into a crisp Kripke model

Definition

Let us assume that $\langle W, R, V \rangle$ is a tree and $w_0$ is its root. The Kripke model $M' = \langle W', R', V' \rangle$ is defined as the one such that

- $W'$ is the same set $W$,
- $R'(w_1, w_2) := \begin{cases} 1, & \text{if } R(w_1, w_2) \neq 0 \\ 0, & \text{if } R(w_1, w_2) = 0 \end{cases}$
- $V'(w, p) := \begin{cases} 1, & \text{if } \eta(w) \leq V(w, p) \\ V(w, p), & \text{otherwise} \end{cases}$

where $w_0, w_1, \ldots, w_n, w$ is the unique path from $w_0$ to $w$, and $\eta(w) = R(w_0, w_1) \land \ldots \land R(w_n, w)$. 
Converting a Kripke model into a crisp one in the Gödel case

A solution for the previous Example

Initial Kripke Model

Unravelling

Final Conversion
Converting a Kripke model into a crisp one in the Gödel case

Theorem

Let us assume that \( \langle W, R, V \rangle \) is a tree and \( w_0 \) is its root. Then,

\[
\forall \phi \text{ modal formula}, \quad V(w_0, \phi) = V'(w_0, \phi).
\]
Converting a Kripke model into a crisp one in the Gödel case

**Theorem**

Let us assume that \( \langle W, R, V \rangle \) is a tree and \( w_0 \) is its root. Then,

for every modal formula \( \varphi \), \( V(w_0, \varphi) = V'(w_0, \varphi) \).

**Sketch of the Proof**

By induction on the modal depth of modal formulas we prove that for every modal formula \( \varphi \), it simultaneously holds for every \( w \in W \) that

- \( \eta(w) \leq V(w, \varphi) \) iff \( 1 \leq V'(w, \varphi) \).
- if \( V(w, \varphi) < \eta(w) \), then \( V(w, \varphi) = V'(w, \varphi) \).
Converting a Kripke model into a crisp one in the Gödel case

**Theorem**

Let us assume that \( \langle W, R, V \rangle \) is a tree and \( w_0 \) is its root. Then, for every modal formula \( \varphi \),

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Then, taking \( w = w_0 \) (we remind that \( \eta(w_0) = 1 \)) we get that for every modal formula \( \varphi \),

\[ V(w_0, \varphi) = V'(w_0, \varphi). \]
Consequences of the previous construction

Theorem (only with □)

Let \( h : \text{Fm} \rightarrow \text{A} \) be a non-modal homomorphism and \( \text{A} \) be a complete BL chain. The following statements are equivalent.

1. \( h \) is arising from an idempotent Kripke model,
2. \( h \) is arising from a crisp Kripke model.
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Theorem (only with $\square$) If $A$ is a complete BL chain, then $\Lambda(\text{IFr}, A) = \Lambda(\text{CFr}, A)$. 
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Theorem (only with ☐) If $A$ is a complete BL chain, then 
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**Informal Interpretation of the Previous Result**

On the BL framework if the normality axioms holds then indeed we are only considering crisp Kripke models.
Some Open Problems

1. How to axiomatize the standard modal logics?

2. How to axiomatize the global consequence relation associated with Fr?

In the classical case this follows from the equivalence

\[ \Gamma \vdash \phi \iff \{ \gamma : \gamma \in \omega, \gamma \in \Gamma \} \vdash \phi \]

3. Is decidability preserved adding the \( \Delta \)?

4. Is there some duality for the algebras arising?

5. Is there some connection with the theory of canonical algebras?

6. Can we characterize which first-order formulas are equivalent to a modal formula?
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