SAT IN MONADIC GÖDEL LOGICS: (UN)DECIDABILITY RESULTS AND APPLICATIONS

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SAT IN MONADIC LOGIC

Satisfiability problem (SAT)

Decide if there is a model of a given formula.

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Monadic classical logic

SAT is decidable and it has the finite model property.

GÖDEL LOGICS

History

- Gödel (1933) – finitely valued logics

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Why Gödel logics?

Relevance logics (Dunn & Meyer, 1971), provability logic of Heyting arithmetic (Visser, 1982), fuzzy logic (Hájek, 1998), logic programming (Lifschitz et al, 2001), Kripke models (Beckmann and Preining, 2009), medical expert systems (- & Vetterlein, 2010), ...

Gödel Logics: Syntax and Semantics

Signature

The language is identical to that of classical logic.

GÖDEL LOGICS: SYNTAX AND SEMANTICS

Fix a (closed) truth value set $\{0,1\} \subseteq V \subseteq [0,1]$

$$\mathcal{I} : \mathrm{Atom} \mapsto V$$

GÖDEL LOGICS: SYNTAX AND SEMANTICS

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$$\mathcal{I} : \text{Atom} \mapsto V$$

Extension of $\mathcal{I}$ to all formulas:

$$\mathcal{I}(A \wedge B) = \min\{\mathcal{I}(A), \mathcal{I}(B)\}$$

$$\mathcal{I}(A \vee B) = \max\{\mathcal{I}(A), \mathcal{I}(B)\}$$

$$\mathcal{I}(A \to B) = \begin{cases} \mathcal{I}(B) & \text{if } \mathcal{I}(A) > \mathcal{I}(B) \\ 1 & \text{if } \mathcal{I}(A) \leq \mathcal{I}(B) \end{cases}$$

$$\mathcal{I}(\forall x A(x)) = \inf\{\mathcal{I}(A(u)) : u \in U\}$$

$$\mathcal{I}(\exists x A(x)) = \sup\{\mathcal{I}(A(u)) : u \in U\}$$

GÖDEL LOGICS: REMARKS

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- The value of a formula under an interpretation depends only on the *relative ordering* and the *topological type* of the truth values of atomic formulas, and not on their specific values.

E.g. $\mathcal{I}(\Phi(A_1, \ldots, A_n)) \in \{\mathcal{I}(A_1), \ldots, \mathcal{I}(A_n), 1, 0\}$

Gödel Logics: the addition of Δ

We can formalize

- $p(x)$ equal to 0 for some x
- $p(x)$ different from 0 for some x

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The addition of Δ

$$\mathcal{I}(\Delta A) = \begin{cases} 1 & \mathcal{I}(A) = 1 \\ 0 & \text{otherwise} \end{cases}$$

(Baaz, 1996)

Talk outline

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- Chapter 1: SAT in monadic Gödel logic with Δ

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- Open problems

SAT IN MONADIC GÖDEL LOGICS WITH Δ

Two cases

- Finite-valued case: : decidable
- Infinite-valued case: : undecidable

SAT IN MONADIC GÖDEL LOGICS WITH Δ

Undecidability result

Classical theory CE of two equivalence relations.

$$A = \mathcal{Q}^* \bigvee_j \bigwedge_k (x_j^k \equiv_i y_j^k)^l, \quad i = 1, 2 \quad \text{and} \quad l = 1, -1$$

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Fact

SAT-CE is not even recursively enumerable (Rogers, 1956).

SAT IN MONADIC GÖDEL LOGICS WITH Δ

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Undecidability of $\mathbf{SAT\text{-}G}_V^\Delta$

$$\mathcal{I}_{\mathrm{CL}}(A) = 1 \iff \mathcal{I}_{\mathrm{G}}(\tau(A)) = 1$$

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Undecidability of **SAT-G$_V^\Delta$**

$$\mathcal{I}_{\mathrm{CL}}(A) = 1 \iff \mathcal{I}_{\mathrm{G}}(\tau(A)) = 1$$

$$\boxed{\sigma(x \equiv_i y) = \Delta(P_i(x) \leftrightarrow P_i(y))}$$

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$$' \Rightarrow ' \quad \mathcal{I}_{\mathrm{G}}(P_i(u)) = \lambda([u]_i)$$

SAT in Monadic Gödel logics

SAT IN MONADIC GÖDEL LOGICS

Cantor-Bendixon rank

For any topological space X let
$X' = \{x \in X : x \text{ is limit point of } X\}$.
Using transfinite recursion we define the *iterated Cantor-Bendixon derivatives* X^α, α ordinal, as follows:

$$X^0 = X \qquad X^{\alpha+1} = (X^\alpha)' \qquad X^\lambda = \bigcap_{\alpha<\lambda} X^\alpha, \lambda \text{ limit ordinal.}$$

For any $x \in X$, we define its (Cantor-Bendixon-)rank
$|x|_{\mathrm{CB}} = \sup\{\alpha : x \in X^\alpha\}$

SAT IN MONADIC GÖDEL LOGICS

Summary of the results

- Decidable:
- Undecidable:

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- Decidable:
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    - $|0|_{\mathrm{CB}} \geq 2$, 3 predicate symbols

MONADIC LOGICS: $|0|_{\mathrm{CB}} = 0$

Theorem

$$\mathbf{SAT\text{-}G}_V = \mathbf{SAT\text{-}CL}$$

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'$\Leftarrow$' Easy

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'$\Rightarrow$' If $A \in \mathbf{SAT\text{-}G}_V$, define $\mathcal{I}_{\mathrm{CL}}$ as follows:

$$\mathcal{I}_{\mathrm{CL}}(A) = \begin{cases} 1 & \mathcal{I}_{\mathrm{G}}(A) > 0 \\ 0 & \text{o.w.} \end{cases}$$

By induction on the complexity of formulas.
Critical case: $\forall x A(x)$ with $\mathcal{I}_{\mathrm{G}}(\forall x A(x)) = 0$.

MONADIC LOGICS: $|0|_{\mathrm{CB}} = 0$

Theorem

$$\mathbf{SAT\text{-}G}_V = \mathbf{SAT\text{-}CL}$$

Applies to:

- finitely valued logics
- ∃-fragment
- prenex fragment
- monadic witnessed

$|0|_{\mathrm{CB}} \geq 1$, TWO PREDICATES AND S

Theorem

If $|0|_{\mathrm{CB}} \geq 1$ in V, there are at least three predicate symbols, one of which is a constant S strictly between 0 and 1, then **SAT-G**$_V$ is undecidable.

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Proof ($A \in \mathrm{CE}$ $\quad \mathcal{I}_{\mathrm{CL}}(A) = 1 \iff \mathcal{I}_{\mathrm{G}}(\tau(A)) = 1$)

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As for Δ, but we have to translate negation too

$$\sigma(x \equiv_i y) = (P_i(x) \leftrightarrow P_i(y))$$

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$$\tau(A) = \sigma(A) \wedge \forall x(P_1(x) \prec S) \wedge \forall x(P_2(x) \prec S)$$

$|0|_{\mathrm{CB}} \geq 2$, THREE PREDICATE SYMBOLS

Theorem

If $|0|_{\mathrm{CB}} \geq 2$ in V and there are at least three predicate symbols, then $\mathbf{SAT\text{-}G}_V$ is undecidable.

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Proof ideas

- '$\Leftarrow$' ($\mathcal{I}_{\mathrm{G}}(\tau(A)) = 1$ implies $\mathcal{I}_{\mathrm{CL}}(A) = 1$)
- '$\Rightarrow$' ($\mathcal{I}_{\mathrm{CL}}(A) = 1$ implies $\mathcal{I}_{\mathrm{G}}(\tau(A)) = 1$)

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  - forcing the third predicate to decrease to 0: $\neg\forall x P(x) \wedge \forall x \neg\neg P(x)$
  - confine interpretations to intervals below $P(u)$
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  - forcing the third predicate to decrease to 0: $\neg\forall x P(x) \wedge \forall x \neg\neg P(x)$
  - confine interpretations to intervals below $P(u)$
- '$\Rightarrow$' ($\mathcal{I}_{\mathrm{CL}}(A) = 1$ implies $\mathcal{I}_{\mathrm{G}}(\tau(A)) = 1$)
  - parallel execution of the above construction for each of these intervals
  - multiplication of the universe for each of these intervals

The translation

$$\sigma_{a,b}(r \equiv_i s) = (P_i(r) \leftrightarrow P_i(s))$$
$$\sigma_{a,b}(r \not\equiv_i s) = ((P_i(r) \leftrightarrow P_i(s)) \rightarrow P(a)))$$

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$$\begin{aligned}
\tau(A) = \neg\forall x P(x) \wedge \forall x \neg\neg P(x) \wedge \\
\forall x (P(x) \vee \exists y \exists z [P(z) \prec P(y) \wedge P(y) \prec P(x) \wedge \\
\forall u (P(u) \to P(z) \vee P(y) \to P(u)) \wedge \\
\exists w (P(z) \prec P_1(w) \prec P(y) \wedge \\
P(z) \prec P_2(w) \prec P(y)) \wedge \sigma_{y,z}(A)])
\end{aligned}$$

THE TRANSLATION

$$\sigma_{a,b}(\forall r B) = \forall r(P_1(r) \prec P(b) \vee P(a) \prec P_1(r) \vee P_2(r) \prec P(b) \vee P(a) \prec P_2(r)$$

$$\sigma_{a,b}(\exists r B) = \exists r((P(b) \prec P_1(r) \prec P(a)) \wedge (P(b) \prec P_2(r) \prec P(a)) \wedge \sigma_{a,b}(B))$$

$$\sigma_{a,b}(\bigvee_j \bigwedge_k (r_j^k \equiv_i s_j^k)^l) = \bigvee_j \bigwedge_k \sigma((r_j^k \equiv_i s_j^k)^l)$$

$$\sigma_{a,b}(r \equiv_i s) = (P_i(r) \leftrightarrow P_i(s))$$

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CHAPTER 3: THE FRAGMENT FO^1_{mon}

Motivation

The formalization of rule-based systems often require the use of a *fragment* of monadic logic.

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Definition

The fragment FO^1_{mon} consists of all closed formulas in the language with Δ of the form

$$\bigvee_{i=1}^{n} (\exists x A^i_1(x) \wedge \ldots \wedge \exists x A^i_{n_i}(x) \wedge \forall x B^i_1(x) \wedge \ldots \wedge \forall x B^i_{m_i}(x))$$

with A^i_k and B^i_k quantifier-free containing no constant symbol.

RESULTS FOR FO^1_{mon}

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1. $|1|_{\text{CB}} = 0$ in V, then **SAT-**FO^1_{mon} is decidable and enjoys the finite model property
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RESULTS FOR FO^1_{mon}

1. $|1|_{\mathrm{CB}} = 0$ in V, then $\mathbf{SAT}$-FO^1_{mon} is decidable and enjoys the finite model property
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Classical negation

Adding an involutive negation ($\mathcal{I}(\sim A) = 1 - \mathcal{I}(A)$) does not change the decidability status

FO^1_{mon}: THE CASE $|1|_{\mathrm{CB}} = 0$

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$$\text{E.g.} \quad \exists x(\neg\neg A(x) \wedge \neg\Delta A(x))$$

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Theorem

If $|1|_{\mathrm{CB}} = 0$ in V, then $\mathbf{SAT\text{-}G}^{\Delta}_{V}$ is decidable for FO^1_{mon}.

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Theorem

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Proof

$$\text{Consider} \quad \exists x A(x) \wedge \forall x B(x)$$

use Skolemization.

FO^1_{mon}: THE CASE $|1|_{\mathrm{CB}} > 0$

Example

$$P := \exists x A(x) \wedge \forall x \neg \Delta A(x)$$

is satisfiable in $\mathbf{G}^{\Delta}_{V}$ but it has no finite model.

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Δ-chains

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Δ-chains

We use $\prec_{\Delta}$ and $\equiv_{\Delta}$

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Δ-chains

Let F be any formula in FO^1_{mon} and $A_1, \ldots, A_n$ be the predicates occurring in F.

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Example

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Δ-chains

Let F be any formula in FO^1_{mon} and $A_1, \ldots, A_n$ be the predicates occurring in F. A Δ-*chain* over F is any formula of the form

$$(\bot \bowtie_0 A_{\pi(1)}(x)) \land (A_{\pi(1)}(x) \bowtie_1 A_{\pi(2)}(x)) \land (A_{\pi(n)}(x) \bowtie_n \top)$$

where π is a permutation of $\{1, \ldots, n\}$, $\bowtie_i$ is either $\prec_\Delta$ or $\equiv_\Delta$, and at least one of the $\bowtie_i$'s stands for $\prec_\Delta$.

FO^1_{mon}: CHAINS CONT.

- every Δ-chain C_i induces an evaluation the predicates of F
- if $\mathcal{C}_F$ is the set of all chains, then $\bigvee_{C \in \mathcal{C}_F} C$ is valid in $\mathbf{G}^{\Delta}_{V}$.

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Syntactic evaluation $P^C_{A(x)}$

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Syntactic evaluation $P^C_{A(x)}$

For every quantifier-free subformula $A(x)$ of F and every Δ-chain C over F there is a predicate symbol (or $\top$ or $\bot$) $P^C_{A(x)}$ such that

$$\mathcal{I}(C \wedge A(x)) = \mathcal{I}(C \wedge P^C_{A(x)})$$

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Consider $\exists x A(x) \wedge \forall x B(x)$.

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Transformation of the existential quantifier

$$\exists x A(x) \equiv \exists x(\bigvee_{C \in \mathcal{C}_F} C \wedge A(x)) \equiv \bigvee_{C \in \mathcal{C}_F} \exists x (C \wedge P^C_{A(x)})$$

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- delete disjuncts with $P^C_{A(x)}$ being $\perp$
- collect the remaining chains in Γ

FO^1_{mon}: THE CASE $|1|_{\mathrm{CB}} > 0$

Transformation of the universal quantifier

$$\begin{aligned}
\forall x B(x) &\overset{\mathrm{SAT}}{\equiv} \Delta\forall x B(x) \overset{\mathrm{SAT}}{\equiv} \forall x \Delta B(x) \\
&\overset{\mathrm{SAT}}{\equiv} \forall x ((\bigvee_{C \in \mathcal{C}_F} C) \wedge \Delta B(x)) \overset{\mathrm{SAT}}{\equiv} \forall x (\bigvee_{C \in \mathcal{C}_F} (C \wedge \Delta B(x))) \\
&\overset{\mathrm{SAT}}{\equiv} \forall x (\bigvee_{C \in \mathcal{C}_F} (C \wedge P^C_{\Delta B(x)})) \\
&\overset{\mathrm{SAT}}{\equiv} \forall x (\bigvee_{C \in \Pi \subseteq \mathcal{C}_F} C)
\end{aligned}$$

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Transformation of the universal quantifier

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&\overset{\mathrm{SAT}}{\equiv} \forall x(\bigvee_{C\in\Pi\subseteq\mathcal{C}_F} C)
\end{aligned}$$

$$\exists x A(x) \wedge \forall x B(x) \equiv_{SAT} \bigvee_{C\in\Gamma} \exists x(C \wedge P^C_{A(x)}) \wedge \forall x(\bigvee_{C\in\Pi} C)$$

FO^1_{mon}: THE CASE $|1|_{\mathrm{CB}} > 0$

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(2) if $i < j$ then for all c, $\varphi(P_i(c)) < \varphi(P_j(c))$.

(3) for $i > 0$ then $\lim_{c \to \infty} \varphi(P_i(c)) = 1$

CONSTRUCTION OF THE MODEL

$$F \equiv_{SAT} \bigvee_{C \in \Gamma} \exists x (C \wedge P^{C}_{A(x)}) \wedge \forall x (\bigvee_{C \in \Pi} C)$$

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E.g. for $V = [0,1]$ an evaluation φ satisfying F can be defined as

$$\varphi(P_i(c)) = 1 - \frac{1}{(c+2)^i} \tag{1}$$

CHAPTER 4: AN APPLICATION

"Fuzzy Logic: from Mathematics to Medical Applications"
(project funded by WWTF, 2008-2012)

CHAPTER 4: AN APPLICATION

CADIAG-2

- Computer Assisted DIAGnosis
- medical expert system based on Fuzzy Logic
- designed at the Medical University of Vienna
- internal medicine
- overall accuracy 90% (cf. Adlassnig et al., 1985)

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CADIAG-2's modus operandi is close to proofs in $\mathsf{G}^{\sim}_{[0,1]}$

i.e. Gödel logic with truth values $[0, 1]$ extended with $\sim$.

An application

Description of CADIAG-2

- $S_1, \ldots, S_{1781}$ - symptoms, $D_1, \ldots, D_{342}$ - diagnoses
- $[0, 1]$ value associated with them
- compound formulas - conjunction, disjunction and negation
- minimum, maximum, $1 - x$
- 20.000 rules - causal relationships

CADIAG-2 RULES

- me - mutual exclusiveness
- c_d - confirming to degree d

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  $\langle \alpha, \beta, 0, 0 \rangle$
- c$_d$ - confirming to degree d

IF α *(positive rheumatoid factor)*
THEN NOT β *(seronegative rheumatoid arthritis)*

CADIAG-2 RULES

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- c$_d$ - confirming to degree d

  $\langle \alpha, \beta, 0.3, 0.5 \rangle$

IF α *(strongly reduced number of thrombocytes)*
THEN β *(systemic lupus erythematosus)*
with degree $d = 0.3$.

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IF α *(strongly reduced number of thrombocytes)*
THEN β *(systemic lupus erythematosus)*
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$$value(\beta) = \min\{value(\alpha), d\}$$

A NUMBER INTERPRETATION

Over 20.000 rules, $\langle \alpha, \beta, soc, foo \rangle$

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Intended meaning (Adlassnig et al. 1986)

$$\mathit{soc} = \frac{\Sigma_a \min\{\alpha(a), \beta(a)\}}{\Sigma_a \alpha(a)}, \quad \mathit{foo} = \frac{\Sigma_a \min\{\alpha(a), \beta(a)\}}{\Sigma_a \beta(a)},$$

Σ_a ranges over a database of patients

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Is there a set of patients that fulfills the rules?

A FORMALIZATION IN $\mathsf{G}^{\sim}_{[0,1]}$

me - mutual exclusiveness

- $R = \langle \alpha, \beta, soc, foo \rangle$,

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$$
\begin{array}{ll}
(c_d) & \exists x(\neg\neg A(x) \wedge \neg\neg B(x)) \wedge \\
 & \exists x(((A(x) \to B(x)) \to B(x)) \wedge \neg\neg \sim B(x)) \wedge \\
 & \exists x(((B(x) \to A(x)) \to A(x)) \wedge \neg\neg \sim A(x))
\end{array}
$$

$$
\begin{array}{ll}
(c_1) & \forall x(A(x) \to B(x)) \wedge \exists x \neg\neg A(x) \wedge \\
 & \exists x(((B(x) \to A(x)) \to A(x)) \wedge \neg\neg \sim A(x))
\end{array}
$$

$$
\begin{array}{ll}
(\mathrm{me}) & \forall x(\neg A(x) \vee \neg B(x)) \wedge \exists x \neg\neg A(x) \wedge \exists x \neg\neg B(x)
\end{array}
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$$
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(\mathrm{oc}) & \forall x(B(x) \to A(x)) \wedge \forall x(A(x) \to B(x)) \wedge \\
 & \exists x \neg\neg A(x)
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$$

On the formalization

Note that

The $\mathsf{G}^{\sim}_{[0,1]}$ formulas formalizing CADIAG-2 belongs to the fragment FO^{1}_{mon}.

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Binary rules

Consider the binary fragment of CADIAG-2

$$\langle \alpha, \beta, soc, foo \rangle$$

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The $\mathsf{G}^{\sim}_{[0,1]}$ formulas formalizing CADIAG-2 belongs to the fragment FO^{1}_{mon}.

Binary rules

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Fact: The $\mathsf{G}^{\sim}_{[0,1]}$ formulas formalizing it are satisfiable if and only if they are satisfiable in classical logic.

CHECK IN CLASSICAL LOGIC

Software used

- *Prover9* and *Mace4* (McCune, University of New Mexico)

10 groups of inconsistencies

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An example

me, IF A
THEN NOT B

ao, IF NOT B
THEN NOT C

c_d, IF A
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with degree 0.99.

A = "Nerves, Chorea Minor"
B = "Reactive arthritis"
C = "Rheumatic fever".

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Inconsistency:
$C = 0$
$C = 0.99$

SUMMARY

SAT in Gödel logics

- with Δ: undecidable when V is infinite
- without Δ:
  - $|0|_{\mathrm{CB}} = 0$: decidable
  - $|0|_{\mathrm{CB}} \geq 1$, 3 predicate symbols one of which is a constant $0 < c < 1$: undecidable
  - $|0|_{\mathrm{CB}} \geq 2$, 3 predicate symbols: undecidable

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TAUT in Gödel logics

- with Δ: undecidable when V is infinite
- without Δ:
  - it is decidable when V is finite
  - it is undecidable if there exists $p < 1 \in V$ such that $\{y \in V \mid y \leq p\}$
  - it is open otherwise (only one case: $V = \{1 - \frac{1}{k} \mid k \geq 1\}$)

OPEN PROBLEMS

(Un)decidability status of SAT in Gödel logics

Applications

OPEN PROBLEMS

(Un)decidability status of SAT in Gödel logics

- monadic, with $|0|_{\mathrm{CB}} = 1$, no special predicate constant (only one case: $V = \{0\} \cup \{\frac{1}{k} \mid k \geq 1\}$)
- monadic, with $|0|_{\mathrm{CB}} \geq 1$ and 1 (or 2) predicate symbols
- one-variable

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Applications

- Check the consistency of the full KB of CADIAG-2
  - SAT solvers and provers for unsatisfiability of FO^1_{mon} in $\mathsf{G}^{\sim}_{[0,1]}$

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