Weakly locally finite MV-algebras and real-valued multisets

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Talk based on joint work with Enzo Marra

MV-algebras

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Boolean algebras coincide with the MV-algebras satisfying the condition

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ℓ-groups

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$$[0, u] := \{x \in G : 0 \leq x \leq u\}$$

with the operations

$$\neg x = u - x,$$

$$x \oplus y = u \wedge (x + y)$$

becomes an MV-algebra, denoted by $\Gamma(G, u)$.

If $\mathcal{G}_S$ denotes the category whose objects are the lattice ordered commutative groups with a distinguished strong order unit, and whose morphisms are the ℓ-group homomorphisms preserving the order units, we have:

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Theorem (Mundici, J. Funct. Anal., 1986)

Γ is a functor which defines a natural equivalence from the category of MV-algebras and homomorphisms onto the category $\mathcal{G}_S$.

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An MV-algebra is simple provided it has no non trivial homomorphic images.

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$\mathbf{L}_m$ is subalgebra of $\mathbf{L}_n$ iff $m - 1$ divides $n - 1$.

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Theorem

Let A be a finite MV-algebra. For each $\chi \in \mathrm{Max}(A)$, there is a unique $n \geq 2$ such that $\chi(A) = L_n$, and

$$A \cong \prod_{\chi \in \mathrm{Max}(A)} \chi(A).$$

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Hence we have a bijective correspondence between finite multisets and isomorphism classes of finite MV-algebras.

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The category $\mathbb{FBA}$ of finite boolean algebras is a subcategory of $\mathbb{FMV}$. By restriction we obtain the well known natural equivalence between the category of finite sets and the $\mathbb{FBA}^{\mathrm{op}}$.

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(iv)There is a set Y such that A is isomorphic to a subalgebra of the MV-algebra $(\mathbf{Q} \cap [0, 1])^Y$ formed by functions of finite range.
(v) A is a limit of a direct system of finite MV-algebras with injective transition morphisms.

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A supernatural number ν is finite iff $\{p \in \boldsymbol{P} \mid \nu(p) \neq 0\}$ is a finite set and $\nu(p) < \infty$ for all $p \in \boldsymbol{P}$.

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The correspondence $n \mapsto \nu_n$ is an order isomorphism from $\mathbf{N_d}$ (the lattice of natural numbers ordered by divisibility) and the sublattice of $\mathbf{G}$ formed by the finite supernatural numbers.

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For each topological space X, the set $Cont(X, \boldsymbol{G})$ of continuous functions from X into $\boldsymbol{G}$, endowed with the pointwise order, is a complete lattice.

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Objects: All pairs (S, σ) such that S is a Stone space and $\sigma\colon S \to \boldsymbol{G}$ is continuous.

Morphisms: Whenever (R, ρ) and (S, σ) are objects in $\mathbb{C}$, then a morphism

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We shall define a functor from the category $\mathbb{LFMV}$ of locally finite MV-algebras and homomorphisms, to the category $\mathbb{C}$.

The functor $\mathcal{M}$

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Associate with S_χ the generalized natural number ν_χ such that for each prime number p,

$$\nu_\chi(p) = \begin{cases} n & \text{if } \frac{1}{p^n} \in S_\chi \text{ and } \frac{1}{p^{n+1}} \notin S_\chi, \\ \\ \infty & \text{if } \frac{1}{p^k} \in S_\chi \text{ for every integer } k \geq 0. \end{cases}$$

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$\mathcal{M}$ is a functor from $\mathbb{LFMV}$ to $\mathbb{C}^{op}$.

Categorical equivalences

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Corollary

The restriction of $\mathcal{M}$ to $\mathbb{MV}_n$ establishes a natural duality between $\mathbb{MV}_n$ and $\mathbb{C}_n$.

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Corollary

The correspondence $\nu \mapsto \Gamma(G_\nu, 1)$ defines an order isomorphism from the lattice G onto the lattice of subalgebras of the MV-algebra $Q \cap [0, 1]$.

Hence the lattice G of supernatural numbers is isomorphic to the lattice of rational subalgebras of the standard MV-algebra $[0, 1]$.

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An element z in a complete lattice L is compact provided that for each subset X of L, if $z \leq \bigvee_{x \in X} x$, then there is a *finite* set $F \subseteq X$ such that $z \leq \bigvee_{x \in F} x$.

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An algebraic lattice is a complete lattice L such that every element in L is a supremum of compact elements.

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The Scott topology on an algebraic lattice L is the topology having as a basis the principal filters $[z) = \{x \in L : z \leq x\}$ for all $z \in K(L)$.

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By universal algebra, it follows that H is an algebraic lattice whose compact elements are the finitely generated subalgebras.

Suppose H is equipped with the Scott topology and let X be a topological space.

A function $f\colon X \to H$ is continuous if and only if given $x \in X$ and $z \in f(x)$, there is an open neighborhood U of x such that $z \in f(y)$ for all $y \in U$.

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We call the objects of $\mathbb{RM}$ real multisets.

Weakly finite MV-algebras

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The following is a crucial result:

Lemma

An MV-algebra A is weakly finite if and only if A is semisimple and $\mathrm{Max}(A)$ is finite.

Weakly locally finite MV-algebras

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Theorem

The following are equivalent conditions for any MV-algebra A.

(i) $A \in \mathbb{LWF}$.

(ii) *There is a boolean space X such that A is isomorphic to a separating subalgebra $\hat{A}$ of the MV-algebra $\mathrm{Cont}(X, [0,1])$ formed by functions of finite range: $\mathrm{range}(\hat{a})$ finite for all $\hat{a} \in \hat{A}$.*

(iii) *There is a boolean space X such that A is isomorphic to a separating subalgebra of $\mathrm{Cont}(X, [0,1]_d)$.*

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It follows that locally weakly finite MV-algebras are hyperarchimedean.

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Theorem

For any set $I \neq \emptyset$ and family $\{A_i\}_{i \in I}$ of algebras in $\mathbb{LWF}$, there exists $P \in \mathbb{LWF}$, together with homomorphisms $\rho_i \colon P \to A_i$, for each $i \in I$, satisfying the following universal property: Given any $B \in \mathbb{LWF}$ and homomorphisms $f_i \colon B \to A_i$, for $i \in I$, there is a unique homomorphism $h \colon B \to P$ making the following diagram commutative, for each $i \in I$:

$$\begin{array}{ccc} B & \xrightarrow{h} & P \\ f_i \searrow & & \swarrow \rho_i \\ & A_i & \end{array}$$

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Theorem

If $\mathrm{F(A)} = (\mathrm{Max(A)}, \sigma_\mathrm{A})$ *for each object A in* $\mathbb{LWF}$*, and* $\mathrm{F(h)} = \mathrm{h}_*$ *for each morphism* $A \xrightarrow{h} B$ *in* $\mathbb{LWF}$*, then* F *is a contravariant functor from* $\mathbb{LWF}$ *into* $\mathbb{RM}$*, i. e., a functor from* $\mathbb{LWF}$ *into* $\mathbb{RM}^{op}$*.*

The functor G from $\mathbb{RM}$ to $\mathbb{LWF}^{op}$

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Define in $E = E_P$ the topology generated by the subbasic sets of the form $(U \times \{r\}) \cap E$, for U open in X and $r \in [0,\ 1]$.

Given an object $P = (X, \sigma)$ of $\mathbb{RM}$, let E_P be the disjoint union of the MV-algebras $\sigma(x)$ for $x \in X$, equipped with the canonical projection into X:

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Then E_P is Hausdorff and π_P is a local homeomorphism.

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then $\neg$ is a continuous operation from E_P into E_P and $\oplus$ is continuous from $\{(e, f) \in E_P \times E_P : \pi_P(e) = \pi_P(f)\}$ into E_P.

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Lemma

For each object $P = (X, \sigma)$ of $\mathbb{RM}$, (E_P, X, π_P) is a global Hausdorff sheaf of simple MV-algebras.

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(Global means that each $e \in E_P$ belongs to the image of a continuous global section.)

MV-algebras of continuous global sections

Recall that a continuous global section is a continuous function $s\colon X \to E_p$ such that $\pi_P(s(x)) = x$ for all $x \in X$.

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Theorem

Let $P = (X, \sigma)$ be an object of $\mathbb{RM}$. Then (E_P, X, π_P) is a global Hausdorff sheaf of subalgebras of the standard MV-algebra, and the MV-algebra $A(E_P)$ is locally weakly finite.

Morphisms

Let $P = (X, \sigma), Q = (Y, \rho)$ be objects in $\mathbb{RM}$ and let $(P, \sigma) \xrightarrow{f} (Q, \rho)$ be a morphism.

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Theorem

If $\mathrm{G}(P) = A(E_P)$ *for each object* $P = (X, \sigma)$ *of* $\mathbb{RM}$*, and if* $\mathrm{G(f)} = \mathrm{f}^*$ *for each morphism* $P = (X, \sigma) \xrightarrow{f} (Y, \rho) = Q$*, then* G *is a contravariant functor from* $\mathbb{RM}$ *into* $\mathbb{LWF}$*, i. e., a functor from* $\mathbb{RM}$ *into* $\mathbb{LWF}^{op}$*.*

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For each $x \in X$, let $\varepsilon_P(x) = \chi_x \lambda_*$, where χ_x is evaluation at x.
ε_P is an isomorphism from (X, σ) onto $(\mathrm{Max}(A(E_P)), \sigma_{A(E_P)}) =$ G(F((X, σ)) in $\mathbb{RM}$.

For each pair A, B of locally weakly finite MV-algebras and each homomorphism $h\colon A \to B$ the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\sigma_A \downarrow & & \downarrow \sigma_B \\
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If $P = (X, \sigma), Q = (Y, \rho)$ are objects in $\mathbb{RM}$ and $(X, \sigma) \xrightarrow{f} (Y, \rho)$ is a morphism, then the following diagram commutes:

$$\begin{array}{ccc} (X, \sigma) & \xrightarrow{f} & (Y, \rho) \\ \varepsilon_P \downarrow & & \downarrow \varepsilon_Q \\ \mathrm{F}(\mathrm{G}((X, \sigma))) & \xrightarrow{\mathrm{F}(\mathrm{G}(f))} & \mathrm{F}(\mathrm{G}((Y, \rho))) \end{array}$$

Finalremarks

There are examples of non-hyperarchimedean semisimple MV-algebras A such that $\sigma_A\colon \mathrm{Max}(A) \to \mathrm{Sub}([0,1])$ is Scott continuos.

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There are also examples of hyperarchimedean MV-algebras A such that $\sigma_A\colon \mathrm{Max}(A) \to \mathrm{Sub}([0,1])$ is not Scott continuos.