Implicational logics vs. order algebraizable logics

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- Petr Cintula and Carles Noguera. Implicational (Semilinear) Logics I: A New Hierarchy, *Archive for Mathematical Logic* 49 (2010) 417-446.
- James G. Raftery. Order algebraizable logics. Submitted.

Outline

Main idea:

- Lindenbaum-Tarski proof of completeness of CPC relies on the equivalence connective.
- $T \nvdash_{\mathrm{CPC}} \varphi$.
- $\langle\alpha,\beta\rangle \in \Omega(T)$ iff $T \vdash_{\mathrm{CPC}} \alpha \leftrightarrow \beta$.
- Consider the quotient $\mathbf{Fm}_{\mathcal{L}}/\Omega(T)$. It is a Boolean algebra.
- $T \not\models_{\mathbf{Fm}_{\mathcal{L}}/\Omega(T)} \varphi$.
- $\langle\alpha,\beta\rangle \in \Omega(T)$ if, and only, if for every formula $\chi(p)$ in one variable, $T \vdash_{\mathrm{CPC}} \chi(\alpha)$ iff $T \vdash_{\mathrm{CPC}} \chi(\beta)$. (The Leibniz congruence)
- Generalization to non-classical logics.

The Leibniz Hierarchy

1. L is called protoalgebraic if Ω is monotone on $Th(\mathrm{L})$, i.e. for every $T_1, T_2 \in Th(\mathrm{L})$, if $T_1 \subseteq T_2$ then $\Omega(T_1) \subseteq \Omega(T_2)$.
2. L is called equivalential if Ω is monotone and commutes with inverse substitutions on $Th(\mathrm{L})$, i.e. for every $T \in Th(\mathrm{L})$ and every $\sigma \in \mathrm{Sub}_{\mathcal{L}}$, $\Omega(\sigma^{-1}[T]) = \sigma^{-1}[\Omega(T)]$.
3. L is called weakly algebraizable if Ω is monotone and injective on $Th(\mathrm{L})$.
4. L is called algebraizable if Ω is monotone and injective and it commutes with inverse substitutions on $Th(\mathrm{L})$.

The role of (definable) equivalencies in the hierarchy

$\mathcal{L}$ propositional language, $\Leftrightarrow(p, q, \overrightarrow{r}) \subseteq \mathrm{Fm}_{\mathcal{L}}$ set of formulae in two variables and (possibly) parameters.

$\varphi \Leftrightarrow \psi$ denotes the set $\bigcup\{\Leftrightarrow(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{\leq\omega}\}$.

The role of (definable) equivalencies in the hierarchy

- A logic S is protoalgebraic iff there is a set of formulae $\Leftrightarrow(p, q, \overrightarrow{r})$ (called *parameterized equivalence set*) such that

  (R) $\vdash_{\mathrm{S}} p \Leftrightarrow p$

  (MP) $p, p \Leftrightarrow q \vdash_{\mathrm{S}} q$

  (Cng) $p \Leftrightarrow q \vdash_{\mathrm{S}} c(s_1 \ldots s_{i-1}, p, \ldots) \Leftrightarrow c(s_1 \ldots s_{i-1}, q, \ldots)$

- A logic S is equivalential iff there is a set of formulae $\Leftrightarrow(p, q)$ (called *equivalence set*) such that

  (R) $\vdash_{\mathrm{S}} p \Leftrightarrow p$

  (MP) $p, p \Leftrightarrow q \vdash_{\mathrm{S}} q$

  (Cng) $p \Leftrightarrow q \vdash_{\mathrm{S}} c(s_1 \ldots s_{i-1}, p, \ldots) \Leftrightarrow c(s_1 \ldots s_{i-1}, q, \ldots)$

(Parameterized) equivalence sets define the Leibniz congruence:

$\langle \varphi, \psi \rangle \in \Omega(T)$ if, and only, $T \vdash_{\mathrm{L}} \varphi \Leftrightarrow \psi$.

An equivalential (protoalgebraic) logic L is (weakly) algebraizable if there exists a set $\mathcal{E}(p)$ of equations in one variable such that

$$\text{(Alg)} \quad p \dashv\vdash_{\mathrm{L}} \bigcup_{\varphi \approx \psi \in \mathcal{E}(p)} \varphi \Leftrightarrow \psi$$

The truth predicate is equationally definable:

given $\langle A, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$, $a \in F$ iff $\varphi(a) = \psi(a)$ for every $\varphi \approx \psi \in \mathcal{E}(p)$.

Algebraizability

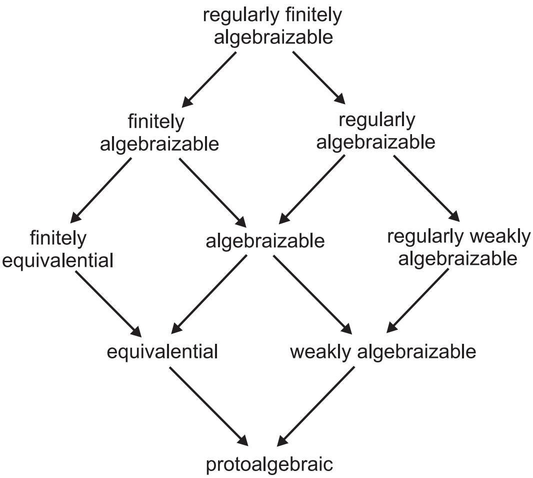
A logic L is algebraizable if it is equivalent to an equational consequence $\models_{\mathcal{K}}$ for some class of algebras $\mathcal{K}$, i.e. there exist translations $\tau : \mathrm{Fm}_{\mathcal{L}} \to \mathrm{Eq}_{\mathcal{L}}$ and $\rho : \mathrm{Eq}_{\mathcal{L}} \to \mathrm{Fm}_{\mathcal{L}}$ such that:

- $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\tau[\Gamma] \models_{\mathcal{K}} \tau(\varphi)$
- $\alpha \approx \beta \models_{\mathcal{K}} \tau[\rho(\alpha \approx \beta)]$ and $\tau[\rho(\alpha \approx \beta)] \models_{\mathcal{K}} \alpha \approx \beta$.

1. L is called *finitely equivalential (algebraizable)* if it is equivalential (algebraizable) with a finite equivalence set.
2. L is called *regularly weakly algebraizable* if it has a parameterized equivalence set satisfying the G-rule.
3. L is called *regularly (finitely) algebraizable* if it has a (finite) equivalence set satisfying the G-rule.

G-rule: $p, q \vdash_{\mathrm{L}} p \Leftrightarrow q$.

Leibniz hierarchy



Outline

1. The common precedent: Leibniz hierarchy in AAL
2. The hierarchy of implicational logics
3. Raftery's order algebraizable logics
4. The synthesis: an enriched hierarchy

A set $\Rightarrow(p, q, \overrightarrow{r}) \subseteq \mathrm{Fm}_{\mathcal{L}}$ is a weak p-implication in a logic L if:

$$
\begin{array}{ll}
(\mathrm{R}) & \vdash_{\mathrm{L}} \varphi \Rightarrow \varphi \\
(\mathrm{MP}) & \varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi \\
(\mathrm{T}) & \varphi \Rightarrow \psi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \Rightarrow \chi \\
(\mathrm{sCng}) & \varphi \Rightarrow \psi, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \chi_{i-1}, \varphi, \ldots, \chi_n) \Rightarrow \\
& \Rightarrow c(\chi_1, \ldots, \chi_{i-1}, \psi, \ldots, \chi_n) \textit{ for each } \langle c, n \rangle \in \mathcal{L} \textit{ and } i \leq n.
\end{array}
$$

We change the prefix *weak* to *algebraic* if there exists a set $\mathcal{E}(p)$ of equations in one variable such that

$$(\mathrm{Alg}) \quad p \dashv\vdash_{\mathrm{L}} \bigcup_{\varphi \approx \psi \in \mathcal{E}(p)} (\varphi \Rightarrow \psi) \cup (\psi \Rightarrow \varphi)$$

We change the prefix *weak* to *regular* if:

$$(\mathrm{Reg}) \quad \varphi, \psi \vdash_{\mathrm{L}} \psi \Rightarrow \varphi$$

We change the prefix *weak* to *Rasiowa* if:

$$(\mathrm{W}) \quad \varphi \vdash_{\mathrm{L}} \psi \Rightarrow \varphi$$

Finally, if $\Rightarrow$ is parameter-free we drop the prefix *p*-.

Lemma

Let L *be a logic and* $\Rightarrow$ *a weak* (p-)implication *in* L*. Then,* L *is equivalential (protoalgebraic) with the (parameterized) equivalence set* $\Leftrightarrow(p, q, \overrightarrow{r}) = \Rightarrow(p, q, \overrightarrow{r}) \cup \Rightarrow(q, p, \overrightarrow{r})$.

Lemma

Each Rasiowa p-implication is a regular p-implication and each regular p-implication is an algebraic p-implication.

The classes of implicational logics

Let L be a logic. We say that L is a

weakly/algebraically/regularly/Rasiowa- (p-)implicational logic

if there is a (parameterized) set of formulae $\Rightarrow$ which is a weak/algebraic/regular/Rasiowa (p-)implication in L.

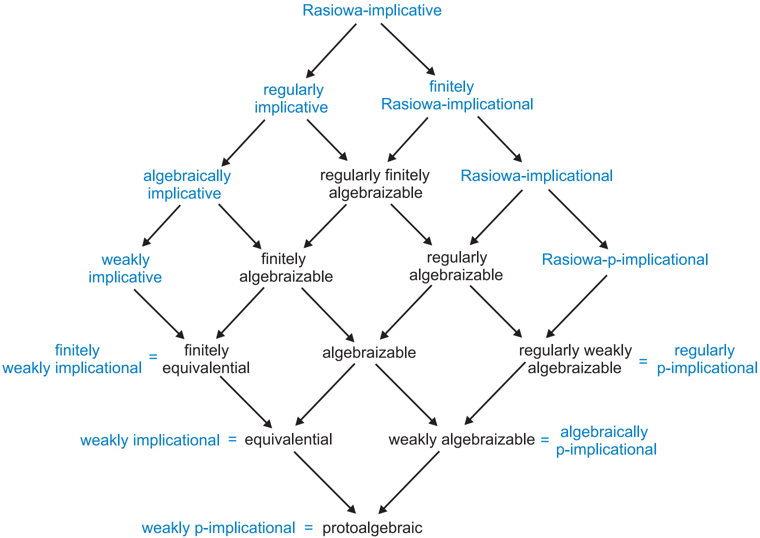
We add the prefix *finitely* if the set $\Rightarrow$ is finite.

We say that L is a

weakly/algebraically/regularly/Rasiowa- implicative logic

if it is a weakly/algebraically/regularly/Rasiowa- implicational logic with $\Rightarrow$ being a parameter-free singleton.

The hierarchy of implicational logics



Implications in the semantics

Definition

Let L be a logic, $\Rightarrow$ a weak p-implication, and $\mathbf{A} = \langle \mathcal{A}, F \rangle$ a matrix. We define a binary relation $\leq_{\mathbf{A}}^{\Rightarrow}$ on A as:

$$a \leq_{\mathbf{A}}^{\Rightarrow} b \qquad \text{iff} \qquad a \Rightarrow^{\mathcal{A}} b \subseteq F$$

Lemma

Let L be a logic, $\Rightarrow$ a weak p-implication, and $\mathbf{A} = \langle \mathcal{A}, F \rangle \in \mathbf{MOD}(\mathrm{L})$. Then:

- $\leq_{\mathbf{A}}^{\Rightarrow}$ *is a preorder.*
- $\leq_{\mathbf{A}}^{\Rightarrow}$ *is an order if, and only if,* $\mathbf{A}$ *is reduced.*
- $\Omega_{\mathcal{A}}(F) = \leq_{\mathbf{A}}^{\Rightarrow} \cap (\leq_{\mathbf{A}}^{\Rightarrow})^{-1}$.
- F *is an upset w.r.t.* $\leq_{\mathbf{A}}^{\Rightarrow}$*, i.e. if* $a \in F$ *and* $a \leq_{\mathbf{F}}^{\Rightarrow} b$*, then* $b \in F$*.*

A formal definition of fuzzy logic

$\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$: class of reduced models where the order is linear.

$\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L}) \subseteq \mathbf{MOD}^{*}(\mathrm{L})_{\mathrm{RFSI}}$.

Definition

Let L be a logic and $\Rightarrow$ a weak p-implication. We say that $\Rightarrow$ is a weak *semilinear* p-implication if $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})}$.

Outline

Semi-equivalence sets

Let $\rho(x, y)$ be a set of binary formulae. Assume that:

$$
\begin{array}{ll}
\text{(R)} & \vdash_{\mathrm{L}} \rho(\varphi, \varphi) \\
\text{(T)} & \rho(\varphi, \psi), \rho(\psi, \chi) \vdash_{\mathrm{L}} \rho(\varphi, \chi) \\
\text{(Subst)} & \rho(\varphi, \psi), \rho(\psi, \varphi), \gamma(\varphi, \overrightarrow{\alpha}) \vdash_{\mathrm{L}} \gamma(\psi, \overrightarrow{\alpha})
\end{array}
$$

Then $\rho(x, y) \cup \rho(y, x)$ is an equivalence set.

$\rho(x, y)$ is called a *semi-equivalence set*.

Implications are semi-equivalencies with *modus ponens*.

Order algebraizability

A logic L is order algebraizable iff there is a set of binary formulae $\rho(x, y)$ and a set $\mathcal{E}(p)$ of pairs of formulae in the variable p such that:

$$
\begin{array}{ll}
(\mathrm{R}) & \vdash_{\mathrm{L}} \rho(\varphi, \varphi) \\
(\mathrm{T}) & \rho(\varphi, \psi), \rho(\psi, \chi) \vdash_{\mathrm{L}} \rho(\varphi, \chi) \\
(\mathrm{Subst}) & \rho(\varphi, \psi), \rho(\psi, \varphi), \gamma(\varphi, \overrightarrow{\alpha}) \vdash_{\mathrm{L}} \gamma(\psi, \overrightarrow{\alpha}) \\
(\mathrm{Alg})' & p \dashv\vdash_{\mathrm{L}} \bigcup_{\langle \varphi, \psi \rangle \in \mathcal{E}(p)} \rho(\varphi, \psi)
\end{array}
$$

Order algebraizability

Partially ordered algebra: $\langle \mathcal{A}, \leq \rangle$.

A logic L is order algebraizable if it is equivalent to an inequational consequence $\models^{\leq}_{\mathcal{K}}$ for some class of partially ordered algebras $\mathcal{K}$, i.e. there exist translations $\tau : \mathrm{Fm}_{\mathcal{L}} \to \mathrm{Ineq}_{\mathcal{L}}$ and $\rho : \mathrm{Ineq}_{\mathcal{L}} \to \mathrm{Fm}_{\mathcal{L}}$ such that:

- $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\tau[\Gamma] \models^{\leq}_{\mathcal{K}} \tau(\varphi)$
- $\alpha \preccurlyeq \beta \models^{\leq}_{\mathcal{K}} \tau[\rho(\alpha \preccurlyeq \beta)]$ and $\tau[\rho(\alpha \preccurlyeq \beta)] \models^{\leq}_{\mathcal{K}} \alpha \preccurlyeq \beta$.

{algebraizable logics} $\subsetneq$ {order algebraizable logics} $\subsetneq$ {equivalential logics}

$\rho(x, y)$ defines a partial order in every $\langle \mathcal{A}, F \rangle \in \mathbf{MOD}^*(\mathrm{L})$:

$$a \leq_F b \text{ iff } \rho^{\mathcal{A}}(a, b) \subseteq F.$$

The ρ-ordered model class is:

$\mathcal{K} = \{\langle \mathcal{A}, \leq_F \rangle \mid \langle \mathcal{A}, F \rangle \in \mathbf{MOD}^*(\mathrm{L})\}$.

Outline

Weak order algebraizability

A logic L is weakly order algebraizable iff there is a parameterized set of formulae $\rho(x, y, \overrightarrow{z})$ and a set $\mathcal{E}(p)$ of pairs of formulae in the variable p such that:

(R)	$\vdash_{\mathrm{L}} \rho(\varphi, \varphi, \overrightarrow{\alpha})$
(T)	$\bigcup_{\overrightarrow{\alpha}} \rho(\varphi, \psi, \overrightarrow{\alpha}), \bigcup_{\overrightarrow{\alpha}} \rho(\psi, \chi, \overrightarrow{\alpha}) \vdash_{\mathrm{L}} \rho(\varphi, \chi, \overrightarrow{\alpha})$
(Subst)	$\bigcup_{\overrightarrow{\alpha}} \rho(\varphi, \psi, \overrightarrow{\alpha}), \bigcup_{\overrightarrow{\alpha}} \rho(\psi, \varphi, \overrightarrow{\alpha}), \gamma(\varphi, \overrightarrow{\beta}) \vdash_{\mathrm{L}} \gamma(\psi, \overrightarrow{\beta})$
(Alg)'	$p \dashv\vdash_{\mathrm{L}} \bigcup_{\langle\varphi,\psi\rangle \in \mathcal{E}(p)} \{\rho(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{\leq\omega}\}$

$\rho(x, y, \overrightarrow{z}) \cup \rho(y, x, \overrightarrow{z})$ is a parameterized equivalence set.

$\rho(x, y, \overrightarrow{z})$ is called a *parameterized semi-equivalence set*.

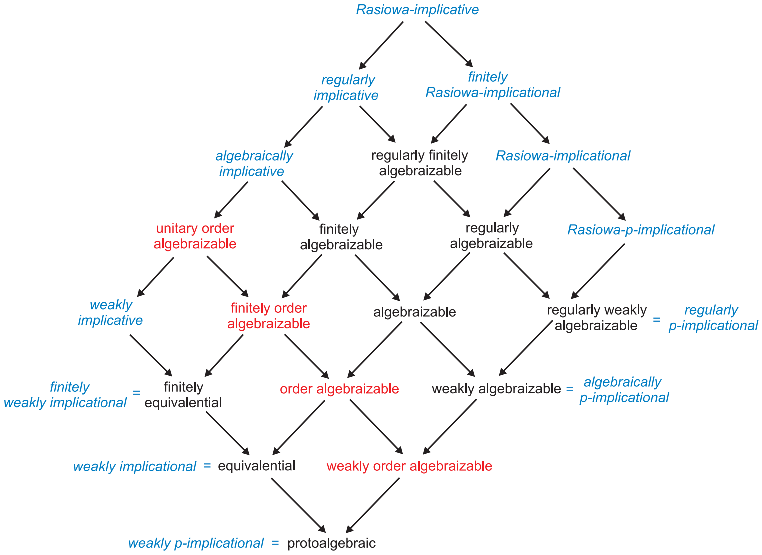
$\rho(x, y, \overrightarrow{z})$ defines a partial order in every $\langle \mathcal{A}, F \rangle \in \mathbf{MOD}^*(\mathrm{L})$:

$$a \leq_F b \text{ iff } \rho^{\mathcal{A}}(a, b, \overrightarrow{c}) \subseteq F \text{ for every } \overrightarrow{c} \in A^{\leq\omega}.$$

The ρ-ordered model class is:

$\mathcal{K} = \{\langle \mathcal{A}, \leq_F \rangle \mid \langle \mathcal{A}, F \rangle \in \mathbf{MOD}^*(\mathrm{L})\}$.

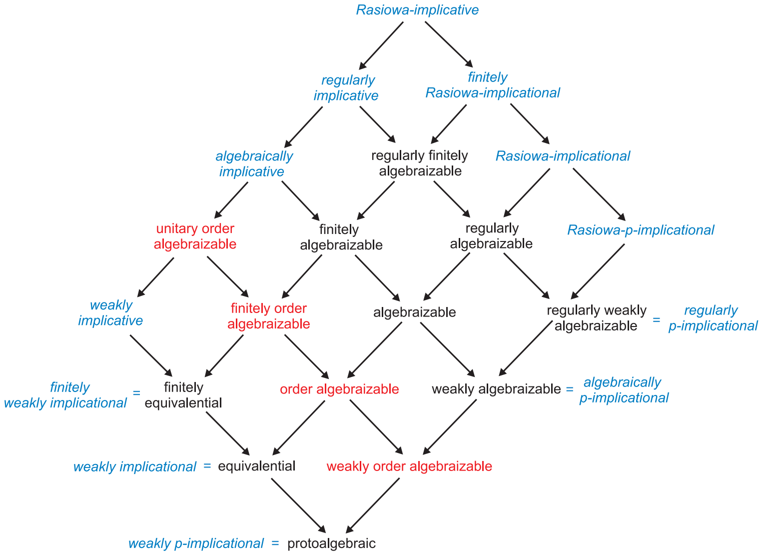
The enriched hierarchy



Separating examples

- BCI is unitary order algebraizable ($\rho(x,y) = \{x \to y\}$, $\tau(p) = \{\langle p \to p, p\rangle\}$) but it is not weakly algebraizable.

The enriched hierarchy



Theorem

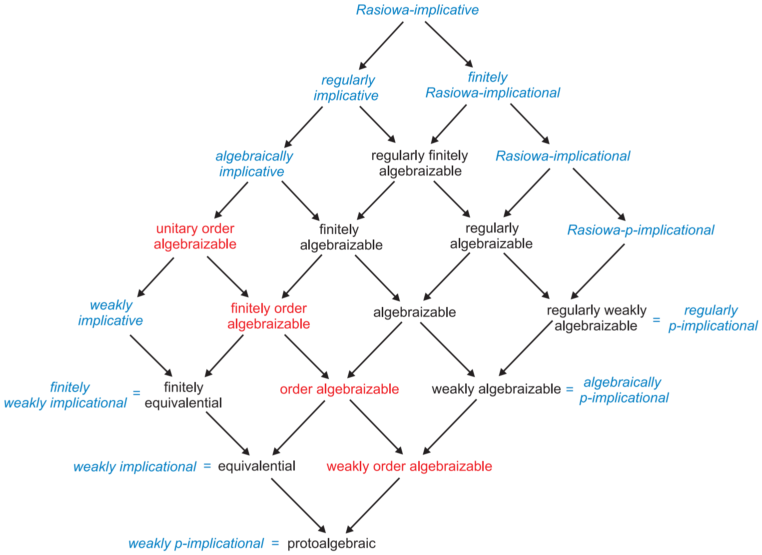
If L *is weakly order algebraizable with* ρ*, then for every algebra* $\mathcal{A}$*:* $F \mapsto \rho^{-1}[F] = \{\langle a, b\rangle \in A^2 \mid \rho^{\mathcal{A}}(a, b, \overrightarrow{c}) \subseteq F$ *for every* $\overrightarrow{c} \in A^{\leq\omega}\}$ *is injective on* $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$*.*

Separating examples

Let E be the Entailment logic of Anderson and Belnap.

- $E_{\rightarrow}$, $E_{\neg,\rightarrow}$ and $E_{\wedge,\rightarrow}$ are weakly implicative but not weakly order algebraizable.

The enriched hierarchy



Conclusion

- Leibniz hierarchy: generalized equivalences and algebraization.
- Hierarchy of implicational logics: generalized implications and algebraization.
- Order algebraizable logics: algebraization with inequations.
- The enriched hierarchy of implicational logics: generalized implications and algebraization with equations or inequations.

Thank you for your attention!

Basic notions

A propositional logic is a pair $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ where $\mathcal{L}$ is a propositional language and $\vdash_{\mathrm{L}}$ is a structural consequence relation.

An $\mathcal{L}$-matrix is a pair $\mathbf{A} = \langle \mathcal{A}, F \rangle$ where $\mathcal{A}$ is an $\mathcal{L}$-algebra and F is a subset of A called the *filter* of $\mathbf{A}$.

The semantical consequence with respect to a class of matrices $\mathbb{K}$ is defined as $\Gamma \models_{\mathbb{K}} \varphi$ iff for each $\mathbf{A} \in \mathbb{K}$ and each $\mathbf{A}$-evaluation e we obtain $e(\varphi) \in F$ whenever $e[\Gamma] \subseteq F$.

$\langle \mathcal{L}, \models_{\mathbb{K}} \rangle$ is a logic.

$\mathbf{A}$ is a model of L if $\vdash_{\mathrm{L}} \subseteq \models_{\mathbf{A}}$.

$\mathbf{MOD}(\mathrm{L})$: the class of all models of L.

$\mathit{Th}(\mathrm{L})$ is the set of all theories of L (sets closed under $\vdash_{\mathrm{L}}$).

Ω can be seen as an operator on $\mathit{Th}(\mathrm{L})$ or, in general, on $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$.

$\mathbf{A} = \langle \mathcal{A}, F \rangle$ is a reduced model if $\Omega_{\mathcal{A}}(F) = \mathit{Id}_{\mathcal{A}}$.

$\mathbf{MOD}^*(\mathrm{L})$: the class of reduced models of L.

Example of an infinite equivalence set

The local consequence defined by Kripke frames, the logic K, is a (non finitely) equivalential logic with:

$$p \Leftrightarrow q = \{p \leftrightarrow q, \Box(p \leftrightarrow q), \Box\Box(p \leftrightarrow q), \ldots\}.$$