Intermediate Fuzzy Quantifiers and Their Properties

Antonín Dvořák, Vilém Novák and Petra Murinová

Institute for Research and Applications of Fuzzy Modeling
University of Ostrava
Czech Republic
web: irafm.osu.cz

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Outline

Motivation for this research

- Elaboration of theory of intermediate quantifiers from Peterson's book Intermediate Quantifiers. Logic, Linguistics and Aristotelian Semantics - analysis of quantifiers *almost all*, *most*, *many*, etc.
- In Peterson's book there is no special formal system and semantics is basically classical.
- Application of fuzzy type theory.

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Quantifiers of type $\langle 1, 1\rangle$

Basic notions for type $\langle 1, 1\rangle$ quantifiers

- Quantifiers of type $\langle 1, 1\rangle$ are denotations of important determiners of NL, e.g.
  - "some" in "Some calendars are Gregorian.",
  - "every" in "Every bird fly.",
  - "many" in "Many students listen to music.", etc.
- A quantifier of type $\langle 1, 1\rangle$ is usually modeled, given a universe M, as a mapping Q_M from $\mathcal{P}(M) \times \mathcal{P}(M)$ to $\{true, false\}$ (or, equivalently, as a binary relation on subsets of M).
- For example, $Q_{1,M}(A, B) = true \iff A \subseteq B$ models every and $Q_{2,M}(A, B) = true \iff A \cap B \neq \emptyset$ models some.
- From the point of view of NL semantics, quantifiers of type $\langle 1, 1\rangle$ are the most important.

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Fuzzy quantifiers

- When we think about generalized quantifiers (*many*, *a few*, ...), we feel that truth values should not change abruptly if we gradually change cardinalities of sets.
- "Many people read books." If the number of people who read books increases by 1, it would be very strange if truth value of this sentence changes from false to true.
- Researchers started to consider more than two truth values, and fuzzy quantifiers emerged, starting from a generalization of the previous definition, where instead of $\{true, false\}$ we consider some other structure of truth values, notably $[0,1]$.

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Quantifiers and measures

- The semantic interpretation of many generalized quantifiers is often connected to measurement of "size" of sets in concern.
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Main ideas

- Formal theory of IQ is a special theory of FTT.
- Intermediate quantifiers are just classical (fuzzy) quantifiers *every* and *some* whose scope is modified using evaluative linguistic expressions, e.g. *very small*, *roughly medium*, etc.
- Thus, intermediate quantifiers are special formulas of FTT consisting of
  - characterization of the size of a given fuzzy set, and
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- Relative simplicity
- Sufficient generality
- Unified definition of IQ
- Possibility to study them by syntactic means

Results

- Peterson: list of 105 generalized Aristotelian syllogisms.
- We were able to prove all of them.
- Syllogisms listed as invalid in Peterson's book are invalid also in our theory.
- We believe that our theory is a reasonable mathematical model of generalized syllogistic.

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Structure of truth values

- The structure of truth values is an MV-algebra with the delta operation

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1}, \Delta \rangle,$$

- or as a special case standard Łukasiewicz$_\Delta$-algebra

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Syntax and Semantics in FTT

Syntax

The (classical) syntax of FTT contains

- definitions of formulas
- 17 axioms
- two inference rules where the rules *modus ponens and generalization* are the rules derivative.

Semantics

- A basic frame for the language J is a system of sets $(M_\alpha)_{\alpha \in Types}$
- A frame is a tuple

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle$$

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Further properties

- theory, interpretation, general model, model, ...

Theorem (Deduction theorem)

Let T be a theory, $A_o \in Form_o$ a formula. Then

$$T \cup \{A_o\} \vdash B_o \text{ iff } T \vdash \Delta A_o \Rightarrow B_o$$

holds for every formula $B_o \in Form_o$.

Theorem (Completeness)

1. *A theory T is consistent iff it has a general model $\mathcal{M}$.*
2. *For every theory T and a formula A_o*

$$T \vdash A_o \text{ iff } T \models A_o.$$

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Trichotomous evaluative linguistic expressions

TEE

- are special expressions of natural language, e.g., *small, big, about fourteen, very short, more or less deep, not thick*.
- Linguistic hedge is an intensifying adverb making the meaning of the evaluative expressions either more, or less specific.
- Hedges can be
  - narrowing — *extremely, significantly, very*
  - widening — *more or less, roughly, quite roughly, very roughly*
  - empty hedge
- Negative TEE

$$\text{not (empty hedge}\langle\text{TE-adjective}\rangle).$$

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Formalization of TEE

- Special theory T^{Ev} of Ł $- FTT$.
- Hedges are represented by formulas $\boldsymbol{\nu} \in Form_{oo}$ whose interpretation is a function $\nu : L \to L$: monotone, sends some truth value to the top and some other truth value to the bottom, and there is an inner truth value b so that $\nu(a) \leq a$ for all $a \leq b$ and $a \leq \nu(a)$ for all $b \leq a$.
- Formulas $Sm\,\boldsymbol{\Delta}$, $Me\,\boldsymbol{\Delta}$, $Bi\,\boldsymbol{\Delta}$ where $\boldsymbol{\Delta}$ are used as a linguistic hedge "utmost" ("limit"). This makes it possible to include classical quantifiers without necessity to introduce them as special cases.

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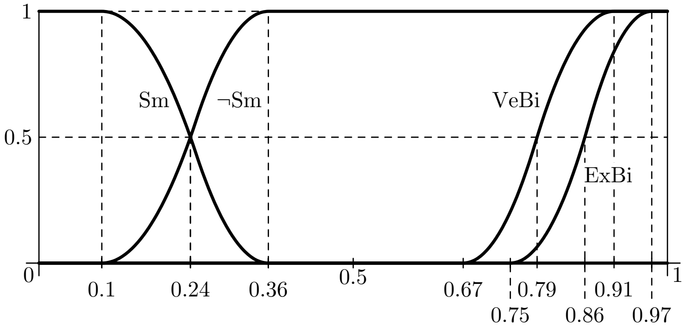
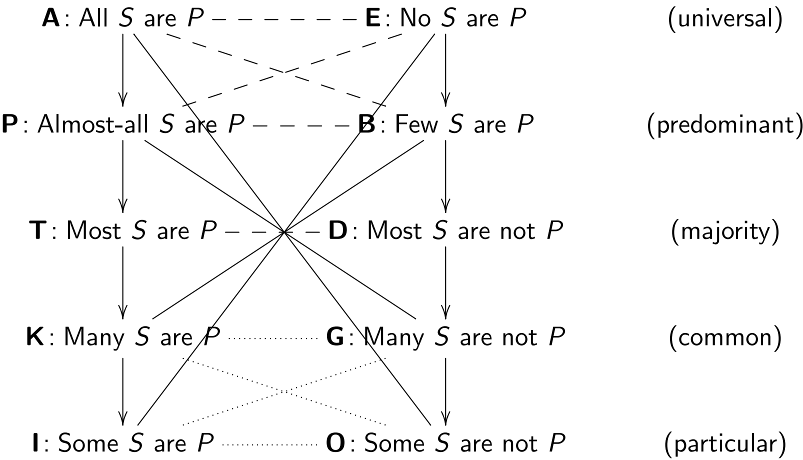
Extensions of some evaluative expressions



Peterson's square



Measure of fuzzy sets

- It is syntactically represented by special formula.
- Its interpretation is a function $M_\alpha \to L$.

Definition

Let $R \in Form_{o(o\alpha)(o\alpha)}$ be a formula. Put $\mu := \lambda z_{o\alpha}\, \lambda x_{o\alpha}\, (Rz_{o\alpha})x_{o\alpha}$. We say that the formula $\mu \in Form_{o(o\alpha)(o\alpha)}$ represents a measure on fuzzy sets in the universe of type $\alpha \in Types$ if

(M1) $\Delta(x_{o\alpha} \equiv z_{o\alpha}) \equiv (\mu z_{o\alpha})x_{o\alpha} \equiv \top)$,

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Definition of T^{IQ}

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Let $\mathcal{S} \subseteq \mathit{Types}$ be a distinguished set of types and

$$\{R \in \mathit{Form}_{o(o\alpha)(o\alpha)} \mid \alpha \in \mathcal{S}\}$$

be a set of new constants. The theory of intermediate quantifiers T^{IQ} w.r.t. $\mathcal{S}$ is a formal theory of Ł-FTT with the language

$$J^{\mathrm{Ev}} \cup \{R_{o(o\alpha)(o\alpha)} \in \mathit{Form}_{o(o\alpha)(o\alpha)} \mid \alpha \in \mathcal{S}\}$$

which is extension of T^{Ev} such that $\mu \in \mathit{Form}_{o(o\alpha)(o\alpha)}$, $\alpha \in \mathcal{S}$, represents a measure on fuzzy sets.

Definition of intermediate quantifiers

Definition

Let T^{IQ} be a theory of intermediate quantifiers and $Ev \in Form_{oo}$ be intension of some evaluative expression. Let $A, B \in Form_{o\alpha}$ be formulas and $z \in Form_{o\alpha}$ and $x \in Form_{\alpha}$ variables where $\alpha \in \mathcal{S}$. Then a type $\langle 1, 1\rangle$ intermediate generalized quantifier interpreting the sentence

"⟨Quantifier⟩ B's are A"

is one of the following formulas:

$$(Q^{\forall}_{Ev}\, x)(B, A) := (\exists z)((\mathbf{\Delta}(z \subseteq B)\,\&\,(\forall x)(z\,x \Rightarrow Ax)) \wedge Ev((\mu B)z)),$$

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IQ with presupposition

Definition

Let T^{IQ} be the theory of intermediate quantifiers and $Ev \in Form_{oo}$ be intension of some evaluative expression. Then an intermediate generalized quantifier with presupposition is represented by the following formula:

$$(Q^{\forall}_{Ev}\, x)(B, A) \equiv (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&\,(\exists x)zx\,\&\,\&\,(\forall x)(z\,x \Rightarrow Ax)) \wedge Ev((\mu B)z)).$$

Remark

Note that only non-empty subsets of B are considered in this definition.

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“All”, “No”, “Almost all”, “Few”, “Most”

A: All B are $A := Q^{\forall}_{Bi\boldsymbol{\Delta}}(B,A) \equiv (\forall x)(Bx \Rightarrow Ax)$,

E: No B are $A := Q^{\forall}_{Bi\boldsymbol{\Delta}}(B,\neg A) \equiv (\forall x)(Bx \Rightarrow \neg Ax)$,

P: Almost all B are $A := Q^{\forall}_{Bi\ Ex}(B,A) \equiv$

$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&\,(\forall x)(zx \Rightarrow Ax)) \wedge (Bi\ Ex)((\mu B)z)),$$

B: Few B are A (:= Almost all B are not A) $:= Q^{\forall}_{Bi\ Ex}(B,\neg A) \equiv$

$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&\,(\forall x)(zx \Rightarrow \neg Ax)) \wedge (Bi\ Ex)((\mu B)z)),$$

T: Most B are $A := Q^{\forall}_{Bi\ Ve}(B,A) \equiv$

$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&\,(\forall x)(zx \Rightarrow Ax)) \wedge (Bi\ Ve)((\mu B)z)),$$

D: Most B are not $A := Q^{\forall}_{Bi\ Ve}(B,\neg A) \equiv$

$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&\,(\forall x)(zx \Rightarrow \neg Ax)) \wedge (Bi\ Ve)((\mu B)z)),$$

"Many", "Some"

K: Many B are $A := Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(B, A) \equiv$

$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\boldsymbol{\&}(\forall x)(zx \Rightarrow Ax)) \wedge \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z)),$$

G: Many B are not $A := Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(B, \neg A) \equiv$

$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\boldsymbol{\&}(\forall x)(zx \Rightarrow \neg Ax)) \wedge \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z)),$$

I: Some B are $A := Q^{\exists}_{Bi\boldsymbol{\Delta}}(B, A) \equiv (\exists x)(Bx \wedge Ax)$,

O: Some B are not $A := Q^{\exists}_{Bi\boldsymbol{\Delta}}(B, \neg A) \equiv (\exists x)(Bx \wedge \neg Ax)$.

Monotonicity

- $T^{\mathrm{IQ}} \vdash (\mathbf{A}) \Rightarrow (\mathbf{P})$, $T^{\mathrm{IQ}} \vdash (\mathbf{P}) \Rightarrow (\mathbf{T})$, $T^{\mathrm{IQ}} \vdash (\mathbf{T}) \Rightarrow (\mathbf{K})$.
- $T^{\mathrm{IQ}} \vdash (\mathbf{E}) \Rightarrow (\mathbf{B})$, $T^{\mathrm{IQ}} \vdash (\mathbf{B}) \Rightarrow (\mathbf{D})$, $T^{\mathrm{IQ}} \vdash (\mathbf{D}) \Rightarrow (\mathbf{G})$.

Syllogism, validity

Definition

- A syllogism denoted by $\langle P_1, P_2, C\rangle$ is a kind of logical argument in which the *conclusion* C is inferred from two *premises* — *major* P_1 and *minor* P_2.
- By intermediate syllogism we mean traditional syllogism where we replace one or more of its formulas with some containing intermediate quantifiers.
- We say that the IS is
  - strongly valid if $T^{IQ} \vdash P_1 \,\&\, P_2 \Rightarrow C$, or equivalently, if $T^{IQ} \vdash P_1 \Rightarrow (P_2 \Rightarrow C)$
  - weakly valid if $T^{IQ} \cup \{P_1, P_2\} \vdash C$

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Classification of IGS

Suppose that Q_1, Q_2, Q_3 are intermediate quantifiers and $X, Y, M \in Form_{o\alpha}$

Figure I	Figure II
Q_1 M is Y	Q_1 Y is M
Q_2 X is M	Q_2 X is M
Q_3 X is Y	Q_3 X is Y

Figure III	Figure IV
Q_1 M is Y	Q_1 Y is M
Q_2 M is X	Q_2 M is X
Q_3 X is Y	Q_3 X is Y

Classical syllogisms

Figure I	**Figure II**	**Figure III**	**Figure IV**
AAA	*EAE*	*AAI*	*AAI*
EAE	*AEE*	*IAI*	*AEE*
AII	*EIO*	*AII*	*IAI*
EIO	*AOO*	*EAO*	*EAO*
AAI	*EAO*	*OAO*	*EIO*
EAO	*AEO*	*EIO*	*AEO*

IGS with "most"

Figure I	Figure II	Figure III	Figure IV
AAT	*AED*	*ATI*	*AED*
ATT	*ADD*	*ETO*	*ETO*
ATI	*ADO*	*TAI*	*TAI*
EAD	*EAD*	*DAO*	
ETD	*ETD*		
ETO	*ATO*		

IGS with “most” and “many”

Figure I	**Figure II**	**Figure III**	**Figure IV**
AAK	*AEG*	*AKI*	*AEG*
ATK	*ADG*	*EKO*	*EKO*
AKI	*AGO*	*KAI*	*KAI*
AKK	*AGG*	*GAO*	
EAG	*EAG*	**TTI**	
ETG	*ETG*	**DTO**	
EKO	*EKO*		
EKG	*EKG*		

IGS with "almost all", "most" and "many", "few"

Figure I	Figure II	Figure III	Figure IV
AAP	*AEB*	*PAI*	*AEB*
APP	*ABB*	*EPO*	*PAI*
APT	*ABD*	*BAO*	*EPO*
APK	*ABG*	*API*	
API	*ABO*	**PPI**	
EAB	*EAB*	**TPI**	
EPB	*EPB*	**KPI**	
EPD	*EPD*	**PTI**	
EPG	*EPG*	**PKI**	
EPO	*EPO*	**BPO**	
		DPO	
		GPO	
		BTO	
		BKO	

General examples of IGS

EAO-I:

No M are Y	$(\forall x)(Mx \Rightarrow \neg Yx)$
All X are M	$(\forall x)(Xx \Rightarrow Mx) \,\&\, (\exists x)Xx$
Some X are not Y	$(\exists x)(Xx \wedge \neg Yx)$

Theorem

The syllogism above is strongly valid.

General examples of IGS

ATI-I:

$$\frac{\begin{array}{l}\text{All } M \text{ are } Y\\ \text{Most } X \text{ are } M\end{array}}{\text{Some } X \text{ are } Y}$$

$$\frac{\begin{array}{l}(\forall x)(Mx \Rightarrow Yx)\\ (\exists z)((\boldsymbol{\Delta}(z \subseteq X)\,\&\,(\exists x)zx\,\&\,(\forall x)(zx \Rightarrow Mx)) \wedge (Bi\ Ve)(\mu(X)z))\end{array}}{(\exists x)(Xx \wedge Yx).}$$

Theorem

The syllogism above is weakly valid.

General examples of IGS

GPO-III:

$$\frac{\begin{array}{l}\text{Many } M \text{ are not } Y\\ \text{Almost all } M \text{ are } X\end{array}}{\text{Some } X \text{ are not } Y}$$

$$\frac{\begin{array}{l}(\exists z)((\boldsymbol{\Delta}(z \subseteq M)\,\&\,(\forall x)(zx \Rightarrow \neg Yx)) \wedge \neg\, Sm(\bar{\boldsymbol{\nu}})(\mu(M)z))\\ (\exists z)((\boldsymbol{\Delta}(z \subseteq M)\,\&\,(\exists x)zx\,\&\,(\forall x)(zx \Rightarrow Xx)) \wedge (Bi\ Ex)(\mu(M)z))\end{array}}{(\exists x)(Xx \wedge \neg Yx).}$$

Theorem

The syllogism above (with presupposition) is strongly valid.

Proof of GPO-III

It holds that

$$\vdash (zx \Rightarrow \neg Yx) \,\&\, (zx \Rightarrow Xx) \Rightarrow (zx \Rightarrow (Xx \wedge \neg Yx))$$

By properties of FTT and quantifiers we get

$$\vdash \{\boldsymbol{\Delta}(z \subseteq M) \,\&\, (\forall x)(zx \Rightarrow \neg Yx) \,\&\, (\forall x)(zx \Rightarrow Xx)\} \Rightarrow \\ ((\boldsymbol{\Delta}(z \subseteq M) \,\&\, (\exists x)zx) \Rightarrow (\exists x)(Xx \wedge \neg Yx)).$$

This implies that

$$\vdash \{\boldsymbol{\Delta}(z \subseteq M) \,\&\, (\forall x)(zx \Rightarrow \neg Yx)\} \Rightarrow \\ \{(\boldsymbol{\Delta}(z \subseteq M) \,\&\, (\exists x)zx \,\&\, (\forall x)(zx \Rightarrow Xx)) \Rightarrow (\exists x)(Xx \wedge \neg Yx)\}.$$

Proof of GPO-III cont.

Using properties of TEV:

$$\vdash ((Bi(\boldsymbol{\nu})(\mu(M)z) \Rightarrow \neg\, Sm(\boldsymbol{\nu})(\mu(M)z)).$$

From this,

$$\vdash (\boldsymbol{\Delta}(z \subseteq M)\,\boldsymbol{\&}(\exists x)zx\,\boldsymbol{\&}(\forall x)(zx \Rightarrow Xx)) \wedge (Bi\ Ex)(\mu(M)z) \Rightarrow \\ \Rightarrow \{((\boldsymbol{\Delta}(z \subseteq M)\,\boldsymbol{\&}(\forall x)(zx \Rightarrow \neg Yx))\wedge \\ \wedge \neg\, Sm(\boldsymbol{\nu})(\mu(M)z)) \Rightarrow (\exists x)(Xx \wedge \neg Yx)\}.$$

Finally we get by generalization wrt. z

$$\vdash (\exists z)\{(\boldsymbol{\Delta}(z \subseteq M)\,\boldsymbol{\&}(\forall x)(zx \Rightarrow \neg Yx)) \wedge \neg\, Sm(\boldsymbol{\nu})(\mu(M)z)\} \Rightarrow \\ (\exists z)\{(\boldsymbol{\Delta}(z \subseteq M)\,\boldsymbol{\&}(\exists x)zx\,\boldsymbol{\&}(\forall x)(zx \Rightarrow Xx)) \wedge (Bi\ Ex)(\mu(M)z)\} \Rightarrow \\ (\exists x)(Xx \wedge \neg Yx)$$

which is just strong validity of GPO-III.

Simple model with finite set

Consider a frame

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_{\mathbf{\Delta}} \rangle$$

$M_o = [0, 1]$ — the support of the standard Łukasiewicz$_\Delta$ algebra.
The fuzzy equality $=_o$ is the Łukasiewicz biresiduation $\leftrightarrow$.
$M_\epsilon = \{u_1, \ldots, u_r\}$ is a finite set with fixed numbering of its elements and $=_\epsilon$ is

$$[u_i =_\epsilon u_j] = (1 - \left(1 \wedge \frac{|i-j|}{s}\right))$$

for some fixed natural number $s \leq r$.

- All logical axioms of $Ł - FTT$ are true in the degree 1 in $\mathcal{M}$.
- $\mathcal{M}$ is nontrivial.

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Simple model with finite set cont.

- $\alpha \in \mathcal{S}$ iff α is a type not containing the type o of truth values. This means that all sets M_α for $\alpha \in \mathcal{S}$ are finite.
- Let A, B be a fuzzy sets on M_α, $\alpha \in \mathcal{S}$. Put

$$|A| = \sum_{u \in \mathrm{Supp}(A)} A(u), \qquad u \in M_\alpha.$$

$$F_R(B)(A) = \begin{cases} 1 & \text{if } B = \emptyset \text{ or } A = B, \\ \frac{|A|}{|B|} & \text{if } B \neq \emptyset \text{ and } A \subseteq B, \\ 0 & \text{otherwise.} \end{cases}$$

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Example of invalid syllogism

TAT-III:

Most bushes in the park are in blossom.
All bushes in the park are perennial.

Most perennial in the park are in blossom.

- Intuitively it should hold that "Some perennial in the park are in blossom" which is just conclusion of the valid syllogism **TAI-III**.
- It can be shown that in the previous finite model, we can find interpretation where premises are true in degree 1, but the truth degree of conclusion is sharply less than 1. Therefore **TAT-III** cannot be proved in T^{IQ}.

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Limited monadic **L**-fuzzy quantifier of $\langle 1, 1\rangle$ type

Definition

Let **L** be a residuated lattice, M be a universe of discourse. A mapping

$$Q_M : \mathcal{F}_{\mathbf{L}}(M) \times \mathcal{F}_{\mathbf{L}}(M) \longrightarrow L$$

is called the monadic **L**-fuzzy quantifier of type $\langle 1, 1\rangle$ limited to M.

Holčapek, M.: Monadic **L**-fuzzy quantifiers of the type $\langle 1^n, 1\rangle$, FS&S 2008.

Semantic properties of IQ

- We can study semantic properties of IQ like permutation invariance, isomorphism invariance, extension, conservativity etc.
- CONS: $Q_M(B, A) = Q_M(B, A \cap B)$. IQ fulfills.
- EXT: $Q_M(B, A) = Q_{M'}(B, A)$ for $M \subseteq M'$. IQ fulfills.
- PI : $Q_M(B, A) = Q_M(f^{\rightarrow}(B), f^{\rightarrow}(A))$ for bijection f. IQ generally does not fulfil. Axioms for for fuzzy measure should be strengthened.

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Fuzzy quantifiers determined by measures

Dvořák and Holčapek introduced fuzzy quantifiers determined by fuzzy measures.

Definition

Let $M \neq \emptyset$, $\mathcal{S}(M)$ is a functional and φ_M be an operation on $\mathcal{F}_{\mathbf{L}}(M)$, e.g., $\varphi(A, B) = A \cap B$ or $\varphi(A, B) = A \to B$. An fuzzy quantifier of type $\langle 1, 1 \rangle$ limited to M determined by $(\mathcal{S}(M), \varphi_M)$

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Fuzzy quantifiers determined by measures cont.

- These FQ determined by fuzzy measures were introduced on semantic level only.
- Various semantic properties are known.
- They can be used after minor modifications as models (or directly as lower approximations) for IQ.
- Vice versa, formal theory of IQ can be used as inspiration for the formalization of FQ determined by fuzzy measures.

Conclusions, further research

- We presented formal theory intermediate quantifiers in FTT.
- Investigation of further semantic properties.
- Applications in time series analysis etc.