Loomis-Sikorski Theorem and Stone Duality Theorems for MV-algebras with Internal State

Anatolij DVUREČENSKIJ

Antonio Di Nola, Ada Lettieri

Mathematical Institute, Slovak Academy of Sciences,
Štefánikova 49, SK-814 73 Bratislava, Slovakia
E-mail: `dvurecen@mat.savba.sk`

The talk given at the Logic, Algebra and Truth Degrees 2010, 7-11 September 2010, Prague, Czech Republic.

State MV-algebras

- State MV-algebras, Flaminio-Montagna

State MV-algebras

- State MV-algebras, Flaminio-Montagna
- MV-algebras with internal state

State MV-algebras

- State MV-algebras, Flaminio-Montagna
- MV-algebras with internal state
- (M, τ) M - MV-algebra, τ internal state

State MV-algebras

- State MV-algebras, Flaminio-Montagna
- MV-algebras with internal state
- (M, τ) M - MV-algebra, τ internal state
- τ state-morphism

State MV-algebras

- State MV-algebras, Flaminio-Montagna
- MV-algebras with internal state
- (M, τ) M - MV-algebra, τ internal state
- τ state-morphism
- variety, subvariety
- $s(a)$ averaging of appearing a, $Pr(a)$ truth value of appearing the event a

State MV-algebras

- State MV-algebras, Flaminio-Montagna
- MV-algebras with internal state
- (M, τ) M - MV-algebra, τ internal state
- τ state-morphism
- variety, subvariety
- $s(a)$ averaging of appearing a, $Pr(a)$ truth value of appearing the event a

- $\tau : M \to M$ state operator: $\tau(1) = 1$,

- $\tau : M \to M$ state operator: $\tau(1) = 1$,
- $\tau(x^*) = \tau(x)^*$,

- $\tau : M \to M$ state operator: $\tau(1) = 1$,
- $\tau(x^*) = \tau(x)^*$,
- $\tau(x \oplus y) = \tau(x) \oplus \tau(y \ominus (x \odot y))$,

- $\tau : M \to M$ state operator: $\tau(1) = 1$,
- $\tau(x^*) = \tau(x)^*$,
- $\tau(x \oplus y) = \tau(x) \oplus \tau(y \ominus (x \odot y))$,
- $\tau(\tau(x) \oplus \tau(y)) = \tau(x) \oplus \tau(y)$.

- $\tau : M \to M$ state operator: $\tau(1) = 1$,
- $\tau(x^*) = \tau(x)^*$,
- $\tau(x \oplus y) = \tau(x) \oplus \tau(y \ominus (x \odot y))$,
- $\tau(\tau(x) \oplus \tau(y)) = \tau(x) \oplus \tau(y)$.
- subdirectly irreducible (M, τ) M is not necessarily linearly ordered,

- $\tau : M \to M$ state operator: $\tau(1) = 1$,
- $\tau(x^*) = \tau(x)^*$,
- $\tau(x \oplus y) = \tau(x) \oplus \tau(y \ominus (x \odot y))$,
- $\tau(\tau(x) \oplus \tau(y)) = \tau(x) \oplus \tau(y)$.
- subdirectly irreducible (M, τ) M is not necessarily linearly ordered,
- $\tau(M)$ is linearly ordered

- $\tau \circ \tau = \tau$

- $\tau \circ \tau = \tau$
- $(\tau(M), \tau)$ state MV-algebra

- $\tau \circ \tau = \tau$
- $(\tau(M), \tau)$ state MV-algebra
- state-morphism operator τ

- $\tau \circ \tau = \tau$
- $(\tau(M), \tau)$ state MV-algebra
- state-morphism operator τ
- τ is an endomorphism s.t. $\tau \circ \tau = \tau$

- $\tau \circ \tau = \tau$
- $(\tau(M), \tau)$ state MV-algebra
- state-morphism operator τ
- τ is an endomorphism s.t. $\tau \circ \tau = \tau$
- state-morphism MV-algebra (M, τ)

Examples

- $A = [0, 1]$ MV-algebra, $M = A \times A$

Examples

- $A = [0,1]$ MV-algebra, $M = A \times A$
- $\tau_1(x,y) = (x,x)$

Examples

- $A = [0,1]$ MV-algebra, $M = A \times A$
- $\tau_1(x,y) = (x,x)$
- $\tau_2(x,y) = (y,y)$

Examples

- $A = [0, 1]$ MV-algebra, $M = A \times A$
- $\tau_1(x, y) = (x, x)$
- $\tau_2(x, y) = (y, y)$
- (M, τ_i), $i = 1, 2$, isomorphic subdirectly irreducible state-morphism MV-algebras, M not linear

Examples

- $A = [0,1]$ MV-algebra, $M = A \times A$
- $\tau_1(x,y) = (x,x)$
- $\tau_2(x,y) = (y,y)$
- (M, τ_i), $i = 1, 2$, isomorphic subdirectly irreducible state-morphism MV-algebras, M not linear
- Question: Does $([0,1]^2, \tau)$ generate the variety of all state-morphism MV-algebras ?

- If G is an Abelian ℓ-group and $M = \Gamma(\mathbb{R} \overrightarrow{\times} G, (1,0))$ or $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (n,0))$, $n \geq 1$, then $\sigma((x,g)) := (x,0)$ is a state-morphism operator on M.

- If G is an Abelian ℓ-group and $M = \Gamma(\mathbb{R} \overrightarrow{\times} G, (1,0))$ or $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (n,0))$, $n \geq 1$, then $\sigma((x,g)) := (x,0)$ is a state-morphism operator on M.
- $A \otimes B$, F be a free MV-algebra generated by $A \times B$.

- If G is an Abelian ℓ-group and $M = \Gamma(\mathbb{R} \overrightarrow{\times} G, (1,0))$ or $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (n,0))$, $n \geq 1$, then $\sigma((x,g)) := (x,0)$ is a state-morphism operator on M.
- $A \otimes B$, F be a free MV-algebra generated by $A \times B$.
- I ideal generated by $d((0,0),0)$, $d((1,1),1)$, $d((x, y_1 \vee y_2), (x,y_1) \vee (x,y_2))$, $d((x, y_1 \wedge y_2), (x,y_1) \wedge (x,y_2))$,

- If G is an Abelian ℓ-group and $M = \Gamma(\mathbb{R} \overrightarrow{\times} G, (1,0))$ or $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (n,0))$, $n \geq 1$, then $\sigma((x,g)) := (x,0)$ is a state-morphism operator on M.
- $A \otimes B$, F be a free MV-algebra generated by $A \times B$.
- I ideal generated by $d((0,0),0)$, $d((1,1),1)$, $d((x, y_1 \vee y_2), (x,y_1) \vee (x,y_2))$, $d((x, y_1 \wedge y_2), (x,y_1) \wedge (x,y_2))$,
- $y_1 + y_2 \in B$ then $d((x,y_1) \odot (x,y_2), 0)$, $d((x, y_1 + y_2), (x,y_1) \oplus (x,y_2))$

- If G is an Abelian ℓ-group and $M = \Gamma(\mathbb{R} \overrightarrow{\times} G, (1,0))$ or $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (n,0))$, $n \geq 1$, then $\sigma((x,g)) := (x,0)$ is a state-morphism operator on M.
- $A \otimes B$, F be a free MV-algebra generated by $A \times B$.
- I ideal generated by $d((0,0),0)$, $d((1,1),1)$, $d((x, y_1 \vee y_2), (x,y_1) \vee (x,y_2))$, $d((x, y_1 \wedge y_2), (x,y_1) \wedge (x,y_2))$,
- $y_1 + y_2 \in B$ then $d((x,y_1) \odot (x,y_2), 0)$, $d((x, y_1 + y_2), (x,y_1) \oplus (x,y_2))$

- $A \otimes B := F/I$

- $A \otimes B := F/I$
- $+$ partial addition: $a + b := a \oplus b$ if $a \odot b = 0$ iff $a \leq b^*$.

- $A \otimes B := F/I$
- $+$ partial addition: $a + b := a \oplus b$ if $a \odot b = 0$ iff $a \leq b^*$.
- s -state on an MV-algebra: $s : M \to [0, 1]$ s.t. $s(a + b) = s(a) + s(b)$ and $s(1) = 1$

- $A \otimes B := F/I$
- $+$ partial addition: $a + b := a \oplus b$ if $a \odot b = 0$ iff $a \leq b^*$.
- s -state on an MV-algebra: $s : M \to [0,1]$ s.t. $s(a+b) = s(a) + s(b)$ and $s(1) = 1$
- $\mathcal{S}(M)$, extremal state $s = \lambda s_1 + (1-\lambda)s_2$, $\lambda \in (0,1)$ then $s = s_1 = s_2$

- $A \otimes B := F/I$
- $+$ partial addition: $a + b := a \oplus b$ if $a \odot b = 0$ iff $a \leq b^*$.
- s -state on an MV-algebra: $s : M \to [0,1]$ s.t. $s(a+b) = s(a) + s(b)$ and $s(1) = 1$
- $\mathcal{S}(M)$, extremal state $s = \lambda s_1 + (1-\lambda)s_2$, $\lambda \in (0,1)$ then $s = s_1 = s_2$
- $\partial_e \mathcal{S}(M)$, Krein-Mil'man: $\mathcal{S}(E) = (ConHul(\partial_e \mathcal{S}(E)))^-$

- $A \otimes B := F/I$
- $+$ partial addition: $a + b := a \oplus b$ if $a \odot b = 0$ iff $a \leq b^*$.
- s -state on an MV-algebra: $s : M \to [0, 1]$ s.t. $s(a + b) = s(a) + s(b)$ and $s(1) = 1$
- $\mathcal{S}(M)$, extremal state $s = \lambda s_1 + (1 - \lambda)s_2$, $\lambda \in (0, 1)$ then $s = s_1 = s_2$
- $\partial_e \mathcal{S}(M)$, Krein-Mil'man: $\mathcal{S}(E) = (ConHul(\partial_e \mathcal{S}(E)))^-$
- $[0, 1] \otimes M$ s state on M

- $\sigma_s : \; [0,1] \otimes M \to [0,1] \otimes M$ by $\sigma_s(\alpha \otimes a) = \alpha s(a) \otimes 1$, $\alpha \in [0,1]$ and $a \in M$. According to σ_s is a state operator on $[0,1] \otimes M$. We recall that $\sigma_s([0,1] \otimes M) = [0,1]$.

- $\sigma_s : [0,1] \otimes M \to [0,1] \otimes M$ by $\sigma_s(\alpha \otimes a) = \alpha s(a) \otimes 1$, $\alpha \in [0,1]$ and $a \in M$. According to σ_s is a state operator on $[0,1] \otimes M$. We recall that $\sigma_s([0,1] \otimes M) = [0,1]$.
- σ_s is an extremal state-morphism iff s is an extremal state

- $\sigma_s : [0,1] \otimes M \to [0,1] \otimes M$ by $\sigma_s(\alpha \otimes a) = \alpha s(a) \otimes 1$, $\alpha \in [0,1]$ and $a \in M$. According to σ_s is a state operator on $[0,1] \otimes M$. We recall that $\sigma_s([0,1] \otimes M) = [0,1]$.
- σ_s is an extremal state-morphism iff s is an extremal state
- otherwise, it is only a state operator

Characterization of S.I.

- **Theorem 0.1** *Let* (M, σ) *be a subdirectly irreducible state-morphism MV-algebra. Then* (M, σ) *is one of the following three possibilities.*

Characterization of S.I.

- **Theorem 0.2** *Let* (M, σ) *be a subdirectly irreducible state-morphism MV-algebra. Then* (M, σ) *is one of the following three possibilities.*

(i) M is linear, $\sigma = \mathrm{id}_M$, and the MV-reduct M is a subdirectly irreducible MV-algebra.

Characterization of S.I.

- **Theorem 0.3** *Let* (M, σ) *be a subdirectly irreducible state-morphism MV-algebra. Then* (M, σ) *is one of the following three possibilities.*

(i) M is linear, $\sigma = \mathrm{id}_M$, and the MV-reduct M is a subdirectly irreducible MV-algebra.

(ii) The state-morphism operator σ is not faithful, M has no nontrivial Boolean elements, and the MV-reduct M of (M, σ) is a local MV-algebra.

- (iii) The state-morphism operator σ is not faithful, M has a nontrivial Boolean element. There are a linearly ordered MV-algebra A, a subdirectly irreducible MV-algebra B, and an injective MV-homomorphism $h:\ A \to B$ such that (M, σ) is isomorphic as a state-morphism MV-algebra with the state-morphism MV-algebra $(A \times B, \sigma_h)$, where $\sigma_h(x, y) = (x, h(x))$ for any $(x, y) \in A \times B$.

-

  (iii) The state-morphism operator σ is not faithful, M has a nontrivial Boolean element. There are a linearly ordered MV-algebra A, a subdirectly irreducible MV-algebra B, and an injective MV-homomorphism $h: A \to B$ such that (M, σ) is isomorphic as a state-morphism MV-algebra with the state-morphism MV-algebra $(A \times B, \sigma_h)$, where $\sigma_h(x, y) = (x, h(x))$ for any $(x, y) \in A \times B$.

- description of subdirectly irreducible state MV-algebras yet unknown

State-morphism BL-algebras

- $\tau : M \to M$ idempotent endomorphism, M BL-algebra

State-morphism BL-algebras

- $\tau : M \to M$ idempotent endomorphism, M BL-algebra
- Every state operator on a linearly ordered BL-algebra is a state-morphism

State-morphism BL-algebras

- $\tau : M \to M$ idempotent endomorphism, M BL-algebra
- Every state operator on a linearly ordered BL-algebra is a state-morphism
- Characterization of S.I. state-morphism BL-algebras is the same as that for state-morphism MV-algebras

- (i) Łukasiewicz: $Ł(x, y) = \max\{x + y - 1, 0\}$ with $x \rightarrow_{Ł} y = \min\{x + y - 1, 1\}$, $\sigma(x) = x$

- (i) Łukasiewicz: $Ł(x, y) = \max\{x + y - 1, 0\}$ with $x \to_{Ł} y = \min\{x + y - 1, 1\}$, $\sigma(x) = x$
- (ii) Gödel: $G(x, y) = \min\{x, y\}$ and $x \to_G y = 1$ if $x \leq y$ otherwise $x \to_G y = y$, $\sigma_a(x) := x$ if $x \leq a$ and $\sigma_a(x) = 1$ otherwise and for $a \in (0, 1]$ let $\sigma^a(x) = x$ if $x < a$ and $\sigma^a(x) = 1$

- (i) Łukasiewicz: $Ł(x, y) = \max\{x + y - 1, 0\}$ with $x \to_{Ł} y = \min\{x + y - 1, 1\}$, $\sigma(x) = x$
- (ii) Gödel: $G(x, y) = \min\{x, y\}$ and $x \to_G y = 1$ if $x \leq y$ otherwise $x \to_G y = y$, $\sigma_a(x) := x$ if $x \leq a$ and $\sigma_a(x) = 1$ otherwise and for $a \in (0, 1]$ let $\sigma^a(x) = x$ if $x < a$ and $\sigma^a(x) = 1$
- (iii) product: $P(x, y) = xy$ and $x \to_P y = 1$ if $x \leq y$ and $x \to_P y = y/x$ $\sigma(x) = x$, $x \in [0, 1]$, or $\sigma(x) = 1$ for any $x > 0$.

Varieties of MV-algebras with Interna

- $\{p_i(x) = q_i(x)\}_{i=1,\dots,n}$ variety W

Varieties of MV-algebras with Interna

- $\{p_i(x) = q_i(x)\}_{i=1,\dots,n}$ variety W
- $\{\tau(p_i(x)) = \tau(q_i(x))\}_{i=1,\dots,n}$,

Varieties of MV-algebras with Interna

- $\{p_i(x) = q_i(x)\}_{i=1,\dots,n}$ variety W
- $\{\tau(p_i(x)) = \tau(q_i(x))\}_{i=1,\dots,n}$,
- $\{p_i(\tau(x)) = q_i(\tau(x))\}_{i=1,\dots,n} \quad (\tau(A) \in W)$,

Varieties of MV-algebras with Interna

- $\{p_i(x) = q_i(x)\}_{i=1,\ldots,n}$ variety W
- $\{\tau(p_i(x)) = \tau(q_i(x))\}_{i=1,\ldots,n}$,
- $\{p_i(\tau(x)) = q_i(\tau(x))\}_{i=1,\ldots,n} \quad (\tau(A) \in W)$,
- $\{\tau(p_i(x)) = q_i(\tau(x))\}_{i=1,\ldots,n}$,

Varieties of MV-algebras with Interna

- $\{p_i(x) = q_i(x)\}_{i=1,\ldots,n}$ variety W
- $\{\tau(p_i(x)) = \tau(q_i(x))\}_{i=1,\ldots,n}$,
- $\{p_i(\tau(x)) = q_i(\tau(x))\}_{i=1,\ldots,n} \quad (\tau(A) \in W)$,
- $\{\tau(p_i(x)) = q_i(\tau(x))\}_{i=1,\ldots,n}$,
- $\{p_i(\tau(x)) = \tau(q_i(x))\}_{i=1,\ldots,n}$.

Varieties of MV-algebras with Interna

- $\{p_i(x) = q_i(x)\}_{i=1,\ldots,n}$ variety W
- $\{\tau(p_i(x)) = \tau(q_i(x))\}_{i=1,\ldots,n}$,
- $\{p_i(\tau(x)) = q_i(\tau(x))\}_{i=1,\ldots,n} \quad (\tau(A) \in W)$,
- $\{\tau(p_i(x)) = q_i(\tau(x))\}_{i=1,\ldots,n}$,
- $\{p_i(\tau(x)) = \tau(q_i(x))\}_{i=1,\ldots,n}$.
- four subvarieties of $\mathcal{SMV}$ ${}^{\tau}W$, W^{τ}, ${}_{\tau}W$ and W_{τ},

- If $W = \mathrm{V}(S_1, \ldots, S_n)$ then

  $$^{\tau}W = W^{\tau} = {}_{\tau}W = W_{\tau}$$

  all coinciding with the subvariety defined by $\tau((nx)^* \wedge x) = 0$, that is, equivalently, $\tau(A) \in W$. Moreover, in this case the result does not depend on the chosen axiomatization of $W = \mathrm{V}(S_1, \ldots, S_n)$,

- If $W = \mathrm{V}(S_1, \ldots, S_n)$ then
$$^{\tau}W = W^{\tau} = {}_{\tau}W = W_{\tau}$$
all coinciding with the subvariety defined by $\tau((nx)^* \wedge x) = 0$, that is, equivalently, $\tau(A) \in W$. Moreover, in this case the result does not depend on the chosen axiomatization of $W = \mathrm{V}(S_1, \ldots, S_n)$,
- Any state MV-algebra with $\tau(A) \in \mathrm{V}(S_1, \ldots, S_n)$ is a state-morphism MV-algebra.

- If $W = \mathrm{V}(S_1, \dots, S_n)$ then
$$ {}^{\tau}W = W^{\tau} = {}_{\tau}W = W_{\tau} $$
all coinciding with the subvariety defined by $\tau((nx)^* \wedge x) = 0$, that is, equivalently, $\tau(A) \in W$. Moreover, in this case the result does not depend on the chosen axiomatization of $W = \mathrm{V}(S_1, \dots, S_n)$,
- Any state MV-algebra with $\tau(A) \in \mathrm{V}(S_1, \dots, S_n)$ is a state-morphism MV-algebra.

- Let (A, τ) be a subdirectly irreducible SMV-algebra in the variety $\mathrm{V}(S_1, \ldots, S_n)^\tau$. Then (A, τ) is one of the following three possibilities:

- Let (A, τ) be a subdirectly irreducible SMV-algebra in the variety $\mathrm{V}(S_1, \ldots, S_n)^\tau$. Then (A, τ) is one of the following three possibilities:

(1) $A = S_r$, $1 \leqslant r \leqslant n$ and τ is the identity

- Let (A, τ) be a subdirectly irreducible SMV-algebra in the variety $\mathrm{V}(S_1, \ldots, S_n)^\tau$. Then (A, τ) is one of the following three possibilities:

(1) $A = S_r$, $1 \leqslant r \leqslant n$ and τ is the identity

(2) the MV-reduct A of (A, τ) is a subdirectly irreducible MV-algebra of rank r, $1 \leqslant r \leqslant n$ and $\tau(x) = 0$, for each $x \in \mathrm{Rad}(A)$.

- (3) There is a subdirectly irreducible MV-algebra D such that (A, τ) is isomorphic, as a state-morphism MV-algebra, to the state-morphism MV-algebra $(S_r \times D, \tau_h)$, with

  $$\tau_h(x, y) = (x, h(x)), \tag{1}$$

  where h is an injective MV-homomorphism from S_r to D.

Stone Duality for Boolean Algebras

- $\mathcal{SB}$ - category; objects are Boolean state MV-algebras (B, τ) and a morphism is a τ-homomorphisms from (B, τ) to (B', τ'), that is, an MV-homomorphism h from B to B' such that
$$h \circ \tau = \tau' \circ h.$$

Stone Duality for Boolean Algebras

- $\mathcal{SB}$ - category; objects are Boolean state MV-algebras (B, τ) and a morphism is a τ-homomorphisms from (B, τ) to (B', τ'), that is, an MV-homomorphism h from B to B' such that
$$h \circ \tau = \tau' \circ h.$$
- category $\mathcal{BSG}$ of *Boolean state spaces*, whose objects are the pairs (Ω, g), where Ω is a Boolean space (= Stone space) and $g : \Omega \to \Omega$ is a continuous map with the property $g \circ g = g$.

- The function

$$\varphi : (B, \tau) \in \mathcal{SB} \mapsto (\Omega(B), g) \in \mathcal{BSG},$$

with $g(F) = \tau^{-1}(F)$ for every $F \in \Omega(B)$, is a contravariant functor from $\mathcal{SB}$ to $\mathcal{BSG}$.

- The function

$$\varphi : (B, \tau) \in \mathcal{SB} \mapsto (\Omega(B), g) \in \mathcal{BSG},$$

  with $g(F) = \tau^{-1}(F)$ for every $F \in \Omega(B)$, is a contravariant functor from $\mathcal{SB}$ to $\mathcal{BSG}$.

- The function

$$\psi : (\Omega, g) \in \mathcal{BSG} \mapsto (B(\Omega), s_g) \in \mathcal{SB},$$

  with $s_g(A) = g^{-1}(A)$ for every $A \in B(\Omega)$, is a contravariant functor from
  $\mathcal{BSG}$ to $\mathcal{SB}$.

- Theorem: The categories $\mathcal{SB}$ and $\mathcal{BSG}$ are dual.

Stone Duality and Bauer Simplices

- Ω is *basically disconnected* provided the closure of every open F_σ subset of Ω is open (an F_σ-set is a countable union of closed sets)

Stone Duality and Bauer Simplices

- Ω is *basically disconnected* provided the closure of every open F_σ subset of Ω is open (an F_σ-set is a countable union of closed sets)
- $\mathcal{DSMV}$ - category of divisible σ-complete state-morphism MV-algebras. Objects- σ-complete state-morphism MV-algebras (A, τ), where A is a σ-complete MV-algebra and τ is a σ-complete endomorphism s.t. $\tau \circ \tau = \tau$. Morphisms from (A, τ) into (A', τ') is any σ-complete MV-homomorphism $h : A \to A'$ such that $h \circ \tau = \tau' \circ h$.

- $\mathcal{BSBS}$ - category of Bauer simplices whose objects are pairs (Ω, g), where $\Omega \neq \emptyset$ is a Bauer simplex such that $\partial_e\Omega$ is basically disconnected, and $g : \Omega \to \Omega$ is an affine continuous function such that $g \circ g = g$, $g : \partial_e\Omega \to \partial_e\Omega$. Morphisms from (Ω, g) into (Ω', g') are continuous affine functions $p : \Omega \to \Omega'$ such that $p : \partial_e\Omega \to \partial_e\Omega'$ and $p \circ g = g' \circ p$.

- $\mathcal{BSBS}$ - category of Bauer simplices whose objects are pairs (Ω, g), where $\Omega \neq \emptyset$ is a Bauer simplex such that $\partial_e\Omega$ is basically disconnected, and $g : \Omega \to \Omega$ is an affine continuous function such that $g \circ g = g$, $g : \partial_e\Omega \to \partial_e\Omega$. Morphisms from (Ω, g) into (Ω', g') are continuous affine functions $p : \Omega \to \Omega'$ such that $p : \partial_e\Omega \to \partial_e\Omega'$ and $p \circ g = g' \circ p$.
- The categories $\mathcal{BSBS}$ and $\mathcal{DSMV}$ are dual.

Loomis–Sikorski Theorem

- Tribe: $\mathcal{T} \subseteq [0,1]^{\Omega}$: (i) $1 \in \mathcal{T}$,

Loomis–Sikorski Theorem

- Tribe: $\mathcal{T} \subseteq [0,1]^{\Omega}$: (i) $1 \in \mathcal{T}$,
- (ii) if $f \in \mathcal{T}$, then $1 - f \in \mathcal{T}$,

Loomis–Sikorski Theorem

- Tribe: $\mathcal{T} \subseteq [0,1]^{\Omega}$: (i) $1 \in \mathcal{T}$,
- (ii) if $f \in \mathcal{T}$, then $1 - f \in \mathcal{T}$,
- (iii) if $f, g \in \mathcal{T}$, then $f \oplus g \in \mathcal{T}$, where $(f \oplus g)(\omega) = \min\{f(\omega) + g(\omega), 1\}$, $\omega \in \Omega$.

Loomis–Sikorski Theorem

- Tribe: $\mathcal{T} \subseteq [0,1]^{\Omega}$: (i) $1 \in \mathcal{T}$,
- (ii) if $f \in \mathcal{T}$, then $1 - f \in \mathcal{T}$,
- (iii) if $f, g \in \mathcal{T}$, then $f \oplus g \in \mathcal{T}$, where $(f \oplus g)(\omega) = \min\{f(\omega) + g(\omega), 1\}$, $\omega \in \Omega$.
- (iv) if $\{f_n\}$ is a sequence of elements from $\mathcal{T}$, then $\bigoplus_n f_n \in \mathcal{T}$, where $(\bigoplus_n f_n)(\omega) := \min\{\sum_n f_n(\omega), 1\}$, $\omega \in \Omega$,

Loomis–Sikorski Theorem

- Tribe: $\mathcal{T} \subseteq [0,1]^{\Omega}$: (i) $1 \in \mathcal{T}$,
- (ii) if $f \in \mathcal{T}$, then $1 - f \in \mathcal{T}$,
- (iii) if $f, g \in \mathcal{T}$, then $f \oplus g \in \mathcal{T}$, where $(f \oplus g)(\omega) = \min\{f(\omega) + g(\omega), 1\}, \omega \in \Omega$.
- (iv) if $\{f_n\}$ is a sequence of elements from $\mathcal{T}$, then $\bigoplus_n f_n \in \mathcal{T}$, where $(\bigoplus_n f_n)(\omega) := \min\{\sum_n f_n(\omega), 1\}, \omega \in \Omega$,
- Loomis-Sikorski Theorem: Mundici, AD

- A σ-complete MV-algebra, $\Omega := \partial_e \mathcal{S}(A)$.
  $a \in A$: $\hat{a} : \partial_e \mathcal{S}(A)\ \hat{a}(s) := s(a)$

- A σ-complete MV-algebra, $\Omega := \partial_e \mathcal{S}(A)$.
  $a \in A$: $\hat{a} : \partial_e \mathcal{S}(A)$ $\hat{a}(s) := s(a)$
- $g : \partial_e \mathcal{S}(A) \to \partial_e \mathcal{S}(A)$ $g(s) := s \circ \tau$ $s \in \partial_e \mathcal{S}(A)$
  is continuous and $g^2 = g$

- A σ-complete MV-algebra, $\Omega := \partial_e \mathcal{S}(A)$.
  $a \in A$: $\hat{a} : \partial_e \mathcal{S}(A)$ $\hat{a}(s) := s(a)$
- $g : \partial_e \mathcal{S}(A) \to \partial_e \mathcal{S}(A)$ $g(s) := s \circ \tau$ $s \in \partial_e \mathcal{S}(A)$ is continuous and $g^2 = g$
- $\mathcal{T}$ -tribe generated by $\hat{A} := \{\hat{a} : a \in A\}$,

- A σ-complete MV-algebra, $\Omega := \partial_e \mathcal{S}(A)$.
  $a \in A$: $\hat{a} : \partial_e \mathcal{S}(A)\ \hat{a}(s) := s(a)$
- $g : \partial_e \mathcal{S}(A) \to \partial_e \mathcal{S}(A)\ g(s) := s \circ \tau\ s \in \partial_e \mathcal{S}(A)$ is continuous and $g^2 = g$
- $\mathcal{T}$ -tribe generated by $\hat{A} := \{\hat{a} : a \in A\}$,
- Then $\tau_g : \mathcal{T} \to \mathcal{T}$ defined by $\tau_g(f) := f \circ g$, $f \in \mathcal{T}$ is state-morphism on $\mathcal{T}$.

-

- **Theorem 0.4 (Loomis–Sikorski Theorem)** *Let* (A, τ) *be a* σ*-complete state-morphism MV-algebra. Then there is a* σ*-complete state-morphism MV-algebra* $(\mathcal{T}, T_g)$*, where* $\mathcal{T}$ *is a tribe of functions from* $[0,1]^\Omega$*, a function* $g : \Omega \to \Omega$ *such that* $g \circ g = g$ *and* $f \circ g \in \mathcal{T}$ *for any* $f \in \mathcal{T}$*, such that* $T_g(f) := f \circ g$, $f \in \mathcal{T}$*, is a* σ*-complete state-morphism-operator on* $\mathcal{T}$*. Moreover, there is a* σ*-state-morphism MV-homomorphism* h *from* $\mathcal{T}$ *onto* A *such that* $h \circ T_g = \tau \circ h$.

- **Theorem 0.5** *Let (A, τ) be a σ-complete state-morphism MV-algebra. Then there is a non-empty basically disconnected compact Hausdorff topological space Ω with a tribe $\mathcal{T}$ of functions from $[0,1]^\Omega$ and with a continuous function $g : \Omega \to \Omega$ such that $g \circ g = g$, and $f \circ g \in \mathcal{T}$ for any $f \in \mathcal{T}$ such $\mathcal{T}_g(f) := f \circ g$, $f \in \mathcal{T}$, is a σ-complete state-morphism-operator on $\mathcal{T}$. Moreover, there is a σ-homomorphism h from $\mathcal{T}$ onto A such that $h \circ \mathcal{T}_g = \tau \circ h$.*

References

- A. Di Nola, A. Dvurečenskij, *State-morphism MV-algebras,* Ann. Pure Appl. Logic **161** (2009), 161–173.

References

- A. Di Nola, A. Dvurečenskij, *State-morphism MV-algebras,* Ann. Pure Appl. Logic **161** (2009), 161–173.
- A. Di Nola, A. Dvurečenskij, A. Lettieri, *Erratum "State-morphism MV-algebras" [Ann. Pure Appl. Logic 161 (2009) 161-173],* Ann. Pure Appl. Logic **161** (2010), 1605–1607.

References

- A. Di Nola, A. Dvurečenskij, *State-morphism MV-algebras,* Ann. Pure Appl. Logic **161** (2009), 161–173.
- A. Di Nola, A. Dvurečenskij, A. Lettieri, *Erratum "State-morphism MV-algebras" [Ann. Pure Appl. Logic 161 (2009) 161-173],* Ann. Pure Appl. Logic **161** (2010), 1605–1607.
- A. Dvurečenskij, *Subdirectly irreducible state-morphism BL-algebras,* Archive Math. Logic

- A. Di Nola, A. Dvurečenskij, A. Lettieri, *On the Loomis–Sikorski theorem for MV-algebras with internal state,*

- A. Di Nola, A. Dvurečenskij, A. Lettieri, *On the Loomis–Sikorski theorem for MV-algebras with internal state,*
- A. Di Nola, A. Dvurečenskij, A. Lettieri, *Stone duality type theorems for MV-algebras with internal state,*

- A. Di Nola, A. Dvurečenskij, A. Lettieri, *On the Loomis–Sikorski theorem for MV-algebras with internal state,*
- A. Di Nola, A. Dvurečenskij, A. Lettieri, *Stone duality type theorems for MV-algebras with internal state,*
- A. Di Nola, A. Dvurečenskij, A. Lettieri, *On varieties of MV-algebras with internal states,* Inter. J. Approx. Reasoning **51** (2010), 680–694.

Thank you for your attention