An advertisement for Kleisli categories

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Long Term Goal

We wish to use José Meseguer's framework of *General Logic* to explore non-classical logics and as a foundation for expert systems.

As part of this work we are trying to generalize this framework in a non-classical (fuzzy) direction.

What is General Logic?

General Logic is intended to allow the description of a wide range of logics in a single categorical framework. The framework consists of a number of definitions that, e.g., axiomatize the model-theoretic entailment of a logic (*institution*), the syntactic entailment (*entailment system*), or its deductive systems (*proof calculus*).

Why Generalize?

Classic General Logic is built on crisp structures and relations.

Evidenced, e.g., by the use of crisp powersets throughout the framework:

$$\vdash_\Sigma \subseteq \mathcal{P}\mathsf{Sen}(\Sigma) \times \mathsf{Sen}(\Sigma)$$

But what if we want a fuzzy set of axioms?

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Our interest is in the management of terms and substitutions in a non-classical setting.

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Recall Monads

Definition (Monad)

A monad **F** on $\mathbb{C}$ is a triple containing a functor $\mathsf{F} : \mathbb{C} \longrightarrow \mathbb{C}$ together with two natural transformations $\mu : \mathsf{F} \circ \mathsf{F} \longrightarrow \mathsf{F}$ and $\eta : \mathrm{id}_{\mathbb{C}} \longrightarrow \mathsf{F}$ such that for each object X:

1. $\mu_X \circ \eta_{\mathsf{F}X} = \mathrm{id}_X$
2. $\mu_X \circ \mathsf{F}(\eta_X) = \mathrm{id}_X$
3. $\mu_X \circ \mu_{\mathsf{F}X} = \mu_X \circ \mathsf{F}(\mu_X)$

A *partially ordered monad* $(\mathsf{F}, \preceq, \eta, \mu)$ has $(\mathsf{F}X, \preceq)$ partially ordered with some order-preserving conditions for μ.

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Example (Power set monad on $\mathtt{Set}$)

The power set monad on $\mathtt{Set}$ is $\mathbf{P} = (\mathsf{P}, \eta, \mu)$:

- $\mathsf{P} : \mathtt{Set} \longrightarrow \mathtt{Set}$ is the covariant power set functor:

$$\mathsf{P}X = \{A \mid A \subseteq X\}$$
$$\mathsf{P}f(A) = \{f(x) \mid x \in A\}$$

- $\eta : \mathrm{id}_{\mathtt{Set}} \longrightarrow \mathsf{P}$ has $\eta_X(x) = \{x\}$
- $\mu : \mathsf{P} \circ \mathsf{P} \longrightarrow \mathsf{P}$ has $\mu_X(\mathcal{A}) = \bigcup \mathcal{A}$

Easily extended to a partially ordered monad $(\mathsf{P}, \subseteq, \eta, \mu)$.

Two More Monads

- The fuzzy power set monad **L** generalizes **P**
- The term monad over a signature Ω, $\mathbf{T}_\Omega = (\mathsf{T}_\Omega, \eta, \mu)$ where
  - $\mathsf{T}_\Omega X$ is the set of terms using X as set of variables
  - η is the injection of variables into terms
  - μ is the canonical isomorphism

Composing Monads

Given monads

$$\mathbf{F} = (\mathsf{F}, \eta^{\mathbf{F}}, \mu^{\mathbf{F}}) \quad \text{and} \quad \mathbf{G} = (\mathsf{G}, \eta^{\mathbf{G}}, \mu^{\mathbf{G}})$$

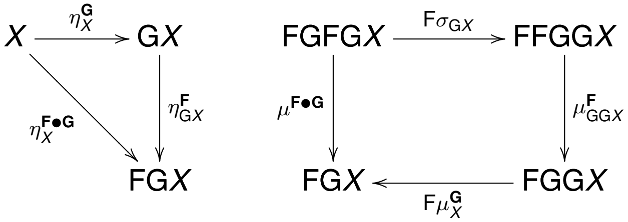
it is sometimes possible to create a *composed monad*

$$\mathbf{F} \bullet \mathbf{G} = (\mathsf{F} \circ \mathsf{G}, \eta^{\mathbf{F} \bullet \mathbf{G}}, \mu^{\mathbf{F} \bullet \mathbf{G}}).$$

Need a natural transformation $\sigma : \mathsf{G} \circ \mathsf{F} \longrightarrow \mathsf{F} \circ \mathsf{G}$ (the swapper) subject to a number of conditions (Beck 1969).

Composing Monads

The conditions on the swapper are precisely what we need to get a composed monad $\mathbf{F} \bullet \mathbf{G} = (\mathsf{F} \circ \mathsf{G}, \eta^{\mathbf{F} \bullet \mathbf{G}}, \mu^{\mathbf{F} \bullet \mathbf{G}})$ where



defines $\eta^{\mathbf{F} \bullet \mathbf{G}}$ and $\mu^{\mathbf{F} \bullet \mathbf{G}}$.

The Monad $\mathbf{L} \bullet \mathbf{T}_\Omega$

It is possible to inductively construct a swapper

$$\sigma : \mathsf{T}_\Omega \circ \mathsf{L} \longrightarrow \mathsf{L} \circ \mathsf{T}_\Omega.$$

Thus, the term and fuzzy power set monads can be composed.

Fuzzy sets of terms!

Recall Kleisli Categories

Definition

The Kleisli category $\mathbb{C}_{\mathbf{F}}$ for a monad $\mathbf{F} = (\mathsf{F}, \eta, \mu)$ on a category $\mathbb{C}$ has objects the same as in $\mathbb{C}$, and morphisms from A to B given by morphisms $f : A \longrightarrow \mathsf{F}B$. Composition of morphisms $f : A \longrightarrow \mathsf{F}B$ and $g : B \longrightarrow \mathsf{F}C$, denoted $f \diamond g$, is given by

$$A \xrightarrow{f \diamond g} \mathsf{F}C = A \xrightarrow{f} \mathsf{F}B \xrightarrow{\mathsf{F}g} \mathsf{F}\mathsf{F}C \xrightarrow{\mu_C} \mathsf{F}C$$

The identity on A is given by $\eta_A : A \longrightarrow \mathsf{F}A$.

Morphisms in $\mathbb{C}_{\mathbf{F}}$ are general *variable substitutions*!

Monad composition and Kleisli

Working with Kleisli categories of composed monads quickly becomes quite complicated and we would like to sweep some of this complexity “under the rug”.

We present an useful construction: *Kleisli over Kleisli*.

The Kleisli Monad

Assume two monads, **F** and **G**, such that $\mathbf{F} \bullet \mathbf{G}$ exists.
We may then define a functor $\mathsf{G}_\mathsf{F} : \mathbb{C}_\mathbf{F} \longrightarrow \mathbb{C}_\mathbf{F}$ such that

$$\mathsf{G}_\mathsf{F} X = \mathsf{G} X$$

and for morphisms $f : X \longrightarrow Y$ in $\mathbb{C}_\mathbf{F}$, let

$$\mathsf{G}_\mathsf{F} f = \sigma_Y \circ \mathsf{G} f.$$

Further, let

$$\eta_X^{\mathbf{G}_\mathbf{F}} = \eta_X^{\mathbf{F} \bullet \mathbf{G}} \quad \text{and} \quad \mu_X^{\mathbf{G}_\mathbf{F}} = \eta_{\mathsf{G} X}^{\mathbf{F}} \circ \mu_X^{\mathbf{G}}.$$

Then $\mathbf{G}_\mathbf{F} = (\mathsf{G}_\mathsf{F}, \eta^{\mathbf{G}_\mathbf{F}}, \mu^{\mathbf{G}_\mathbf{F}})$ is a monad over $\mathbb{C}_\mathbf{F}$. Shown with a great deal of diagram chasing.

Kleisli over Kleisli

Consider now the Kleisli categories of $\mathbf{G}_{\mathbf{F}}$ and $\mathbf{F} \bullet \mathbf{G}$, denoted $[\mathbb{C}_{\mathbf{F}}]_{\mathbf{G}}$ and $\mathbb{C}_{\mathbf{F}\bullet\mathbf{G}}$, respectively.

We find that

$$\mathrm{Ob}([\mathbb{C}_{\mathbf{F}}]_{\mathbf{G}}) = \mathrm{Ob}(\mathbb{C}) = \mathrm{Ob}(\mathbb{C}_{\mathbf{F}\bullet\mathbf{G}})$$

and

$$\begin{aligned}
\mathrm{Hom}_{[\mathbb{C}_{\mathbf{F}}]_{\mathbf{G}}}(X, Y) &= \mathrm{Hom}_{\mathbb{C}_{\mathbf{F}}}(X, \mathrm{G}_{\mathrm{F}}\, Y) \\
&= \mathrm{Hom}_{\mathbb{C}_{\mathbf{F}}}(X, \mathrm{G}\, Y) \\
&= \mathrm{Hom}_{\mathbb{C}}(X, \mathrm{F} \circ \mathrm{G}\, Y) \\
&= \mathrm{Hom}_{\mathbb{C}_{\mathbf{F}\bullet\mathbf{G}}}(X, Y)
\end{aligned}$$

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So, we have

$$[\mathbb{C}_{\mathbf{F}}]_{\mathbf{G}} = \mathbb{C}_{\mathbf{F} \bullet \mathbf{G}}$$

We work with an “atomic” monad on a “composed” category rather than a composed monad on an “atomic” category.

Remains to show whether this easily extend to

Kleisli over Kleisli over Kleisli over . . .

For what might Kleisli over Kleisli be useful?

General Logic

Or rather *Generalized* General Logic

This is where we want to perform substitution . . .

. . . and this is where partially ordered monads come into play.

Please see

Eklund, P. and Helgesson, R., *Monadic extensions of institutions*. Fuzzy Sets Syst. 161, 18 (Sep. 2010), 2354-2368.

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A generalized entailment system, $\mathscr{E}$, is a structure $\mathscr{E} = (\mathtt{Sign}, \mathrm{Sen}, \Phi, \preceq, \eta, \mu, \vdash)$ where

- $\mathtt{Sign}$ is a category of signatures;
- Sen is a functor $\mathrm{Sen} : \mathtt{Sign} \longrightarrow \mathtt{Set}$ taking signatures to sentences;
- $(\Phi, \preceq, \eta, \mu)$ is a partially ordered monad over $\mathtt{Set}$; and
- $\vdash$ is a family of L-valued relations consisting of

$$\vdash_\Sigma : \Phi\mathrm{Sen}(\Sigma) \times \Phi\mathrm{Sen}(\Sigma) \longrightarrow L$$

for each signature $\Sigma \in \mathrm{Ob}(\mathtt{Sign})$ where $\vdash_\Sigma$ is called a Σ-*entailment*.

These are subject to the condition that, assuming $\Gamma, \Gamma', \Gamma'' \in \Phi\mathrm{Sen}(\Sigma)$, each $\vdash_\Sigma$

1. is reflexive, that is, $(\Gamma \vdash_\Sigma \Gamma) = 1$;
2. is *axiom monotone*, that is, $((\Gamma \vee \Gamma') \vdash_\Sigma \Gamma'') \geq (\Gamma \vdash_\Sigma \Gamma'') \vee (\Gamma' \vdash_\Sigma \Gamma'')$;
3. is *consequent invariant*, i.e., $(\Gamma \vdash_\Sigma \Gamma') \wedge (\Gamma \vdash_\Sigma \Gamma'') = (\Gamma \vdash_\Sigma (\Gamma' \vee \Gamma''))$;
4. is transitive, that is, $(\Gamma \vdash_\Sigma \Gamma') \wedge ((\Gamma \vee \Gamma') \vdash_\Sigma \Gamma'') \leq (\Gamma \vdash_\Sigma \Gamma'')$; and
5. is an $\vdash$-*translation*, meaning that $(\Gamma \vdash_\Sigma \Gamma') \leq (\Phi\mathrm{Sen}(\varsigma)(\Gamma) \vdash_{\Sigma'} \Phi\mathrm{Sen}(\varsigma)(\Gamma'))$ for all $\varsigma \in \mathrm{Hom}_{\mathtt{Sign}}(\Sigma, \Sigma')$.

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Generalized institutions can be defined. Logics, theories, proof calculi, etc. can be defined.

Perhaps more interesting, morphisms can be defined for all of these.

On-going work . . .

Definition (Kleene Monad)

A partially ordered monad $\mathbf{F} = (\mathsf{F}, \preceq, \eta, \mu)$ over $\mathtt{Set}$ is said to be a *Kleene monad*, if there exists a natural transformation $0 : \mathrm{id} \longrightarrow \mathsf{F}$ such that the conditions

$$f \diamond 0_X = 0_X$$
$$0_X \diamond f = 0_X$$

are fulfilled for any morphism $f : X \longrightarrow \mathsf{F}X$.

$\mathbf{L}$ is a Kleene monad with $0_X : X \longrightarrow \mathsf{L}X$:

$$0_X(x)(x') = 0$$

for all $x, x' \in X$.

Kleene Monads give Kleene Algebras

Assuming some Kleene monad $\mathbf{F} = (\mathsf{F}, \preceq, \eta, \mu)$ then we may construct a Kleene algebra over each $Hom(X, \mathsf{F}X)$.

Kleene Monads give Kleene Algebras

Let $f, g \in Hom(X, \mathsf{F}X)$, then we have that

$$
\begin{aligned}
0 &= 0_X \\
1 &= \eta_X \\
f + g &= f \vee g \\
f \cdot g &= f \diamond g \\
f^* &= \bigvee_{k=0}^{\infty} f^k
\end{aligned}
$$

See

P. Eklund, R. Helgesson, *Composing partially ordered monads*, 11th International Conference on Relational Methods in Computer Science, RelMiCS11, (Eds. R.Berghammer, A. Jaoua, B. Möller), Lecture Notes in Computer Science **5827** (2009), 88-102

Conclusions

- Kleisli over Kleisli alleviates the complexity of working with Kleisli categories of composed monad
- currently we see its worth when working General Logic and Kleene monads.

Thank you!