Gödel Logics with an Operator that Shifts Truth Values

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2010-09-08

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Overview

- Introduction on propositional and first-order Gödel logic
    - Definitions; Validity; Axiomatization
- Motivation
    - Scarpellini's Theorem for Łukasiewicz logic
- Results

Introduction

Propositional Gödel Logic: Semantics

Definition: Language of propositional Gödel logics $\mathcal{L}^{\mathrm{prop}}$

- a countably infinite set Var of propositional variables
- connectives $\bot$, $\supset$, $\wedge$, $\vee$ (usual arities)

Definition: Propositional Gödel logics for a truth value set $[0,1]$

- Semantics: interpretation $\mathfrak{I}$: Var $\rightarrow [0,1]$

$$\begin{aligned}
\mathfrak{I}(\bot) &:= 0, \\
\mathfrak{I}(A \wedge B) &:= \min\{\mathfrak{I}(A), \mathfrak{I}(B)\}, \\
\mathfrak{I}(A \vee B) &:= \max\{\mathfrak{I}(A), \mathfrak{I}(B)\}, \\
\mathfrak{I}(A \supset B) &:= \begin{cases} 1 & \mathfrak{I}(A) \leq \mathfrak{I}(B), \\ \mathfrak{I}(B) & \mathfrak{I}(A) > \mathfrak{I}(B) \end{cases}
\end{aligned}$$

Propositional Gödel Logic: Abbreviations

Convenient abbreviations:

$$\top := \bot \supset \bot \qquad \mathfrak{I}(\top) = 1$$

$$\neg A := A \supset \bot \qquad \mathfrak{I}(\neg A) = \begin{cases} 1 & \mathfrak{I}(A) = 0, \\ 0 & \mathfrak{I}(A) > 0 \end{cases}$$

$$A \prec B := (B \supset A) \supset B \qquad \mathfrak{I}(A \prec B) = \begin{cases} 1 & \mathfrak{I}(A) < \mathfrak{I}(B) \\ \mathfrak{I}(B) & \mathfrak{I}(A) \geq \mathfrak{I}(B), \end{cases}$$

$$A \leftrightarrow B := (A \supset B) \wedge (B \supset A)$$

$$\mathfrak{I}(A \leftrightarrow B) = \begin{cases} 1 & \mathfrak{I}(A) = \mathfrak{I}(B) \\ \min\{\mathfrak{I}(A), \mathfrak{I}(B)\} & \mathfrak{I}(A) \neq \mathfrak{I}(B) \end{cases}$$

Propositional Gödel Logic: Remarks

- notation: $\supset$ is understood to be right-associative.
- $\bot$, $\supset$, $\wedge$, $\vee$ "project" truth values:
  $$\mathfrak{I}(F(A_1, \ldots, A_n)) \in \{0, \mathfrak{I}(A_1), \ldots, \mathfrak{I}(A_n), 1\}$$
- $\supset$ is the residuum of $\wedge$.
- $\mathfrak{I}(\neg A) = 1$ if and only if $\mathfrak{I}(A) = 0$
- $\mathfrak{I}(\neg\neg A) = \begin{cases} 0 & \mathfrak{I}(A) = 0, \\ 1 & \mathfrak{I}(A) > 0 \end{cases}$
- $\mathfrak{I}(A \supset B) = 1$ if and only if $\mathfrak{I}(A) \leq \mathfrak{I}(B)$
- $\mathfrak{I}(A \prec B) = 1$ if and only if $\mathfrak{I}(A) < \mathfrak{I}(B)$ or $\mathfrak{I}(A) = \mathfrak{I}(B) = 1$
- $\mathfrak{I}(A \leftrightarrow B) = 1$ if and only if $\mathfrak{I}(A) = \mathfrak{I}(B)$
- $A \vee B$ could be defined by $((A \supset B) \supset B) \wedge ((B \supset A) \supset A)$

First-order Gödel Logic: Definitions

Definition:

Interpretation $\mathfrak{I}$ of first-order formulas consists of:

- domain $|\mathfrak{I}|$,
- $P^{\mathfrak{I}}$: $|\mathfrak{I}|^n \rightarrow [0,1]$ for each n-ary predicate symbol P
- $f^{\mathfrak{I}}$: $|\mathfrak{I}|^n \rightarrow |\mathfrak{I}|$ for each n-ary function symbol f

Semantics:

$$\begin{aligned}\mathfrak{I}(\forall x\, A(x)) &:= \inf\{\mathfrak{I}(A(u));\ u \in |\mathfrak{I}|\},\\ \mathfrak{I}(\exists x\, A(x)) &:= \sup\{\mathfrak{I}(A(u));\ u \in |\mathfrak{I}|\}.\end{aligned}$$

Propositional Gödel Logic: Axiomatization

- [Dummett, 1959]
  The set of valid formulas is axiomatized by

○ Intuitionistic prop. logic IPL (MP + a finite set of axiom schemata)
○ + Linearity $(A \supset B) \vee (B \supset A)$

(for full list, see next page)

- Alternatively one can use

○ Hájek's Basic Logic (MP + a finite set of axiom schemata)
○ + $A \supset (A \wedge A)$

Propositional Gödel Logic: Axiomatization

[Dummett, 1959]:
Axiomatization of all valid formulas in $\mathcal{L}^{\mathrm{prop}}$ for $V = [0, 1]$:

MP	$\dfrac{A \quad A \supset B}{B}$
LIN	$(A \supset B) \vee (B \supset A)$
IPL1	$\bot \supset A$
IPL2	$(A \wedge B) \supset A$
IPL3	$(A \wedge B) \supset B$
IPL4	$A \supset B \supset (A \wedge B)$
IPL5	$A \supset (A \vee B)$
IPL6	$B \supset (A \vee B)$
IPL7	$(A \supset C) \supset (B \supset C) \supset (A \vee B) \supset C$
IPL8	$A \supset B \supset A$
IPL9	$(A \supset B \supset C) \supset (A \supset B) \supset (A \supset C)$

First-order Gödel Logic: Axiomatization

Let **H** denote the following Hilbert-type proof system:

IPL

$+ \dfrac{B \supset A(a)}{B \supset \forall x A(x)}, \quad \dfrac{A(a) \supset B}{\exists x A(x) \supset B}$, where a is not free in B,

$+ \forall x A(x) \supset A(t), \quad A(t) \supset \exists x A(x)$

$+$ LIN

$+ \forall x (B \vee A(x)) \supset (B \vee \forall x A(x))$, where x is not free in B.

[Horn, 1969]:

H is sound and complete for first-order Gödel logic.

Motivation

First-order Łukasiewicz logics

Are there connections of (un-)decidability results between fragments of Gödel logic and fragments of Łukasiewicz logics?

[Scarpellini, 1962]:
The set of first-order validities in Łukasiewicz logic is **not** r.e.

Proof idea:

There is an effective embedding t of
classical logic into Łukasiewicz logic s. t.:
For any first-order formula A, the following statements are equivalent:
(1) There is a classical interpret. with finite domain that satisfies A.
(2) There is a Łukasiewicz interpret. $\mathfrak{I}$ such that $0 < \mathfrak{I}(t(A))$.
Makes use of measuring distances and of compactness of [0,1].
Use the Trakhtenbrot theorem to obtain that
$\{A;\ \mathfrak{I}(A) = 0$ for all Ł-interpretations $\mathfrak{I}\}$ is not r.e.

First-order Łukasiewicz logics

Are there connections of (un-)decidability results between fragments of Gödel logic and fragments of Łukasiewicz logics?

Scarpellini's idea does not apply (in an obvious way)
to Gödel logic because distances cannot be expressed.

Try with a new operator such that distances can be measured.

Results

Prop. Gödel Logic + $\bigcirc$: Semantics

Extend the language of propositional Gödel logics $\mathcal{L}^{\mathrm{prop}}$
by a unary connective $\bigcirc$ to obtain $\mathcal{L}^{\mathrm{prop}}_{\bigcirc}$.

Semantics:

Let $r \in [0,1]$. $\mathfrak{I}\colon \mathsf{Var} \to [0,1]$ is an r-**interpretation** if

$$\mathfrak{I}(\bigcirc(A)) = \min\{1, r + \mathfrak{I}(A)\}$$

and $\mathfrak{I}$ is a Gödel interpretation.

Definitions:

A is **valid** if for all $r \in [0,1]$ and all r-interpretations $\mathfrak{I}$: $\mathfrak{I}(A) = 1$.

- Clearly, validity is decidable.

Prop. Gödel Logic + $\bigcirc$: Axiomatization (1/3)

- Neither the 1-entailment nor the entailment relation is compact.
  Take $R := \{\bigcirc^k x \supset y;\ k \in \mathbb{N}\}$, $S := \{y \vee \neg\bigcirc\bot\}$
  so that $R \Vdash S$, but $E \not\Vdash S$ for any finite $E \subseteq R$.
- Usual Lindenbaum method is not applicable.

- Axiomatization is possible with a finite set of axiom schemata:
  IPL + LIN
  $(\bot \leftrightarrow \bigcirc\bot) \supset (A \leftrightarrow \bigcirc A)$,
  $(\bot \prec \bigcirc\bot) \supset (A \prec \bigcirc A)$,
  $\bigcirc(A \supset B) \leftrightarrow (\bigcirc A \supset \bigcirc B)$,

- Algorithm to construct derivation of any valid formula.

Proof idea as follows:

Prop. Gödel Logic + $\bigcirc$: Axiomatization (2/3)

Easy ingredients of the proof:

- $\wedge$, $\vee$, $\supset$ are projecting.
- $\bigcirc(A \Box B) \leftrightarrow (\bigcirc A \Box \bigcirc B)$ for all $\Box \in \{\supset, \leftrightarrow, \prec\}$
- case distinction: $\dfrac{(A \prec B) \supset E, \ (A \leftrightarrow B) \supset E, \ (B \prec A) \supset E}{E}$
- $(A \leftrightarrow B) \supset (C[A] \leftrightarrow C[B])$ for any context C
- in-depth evaluation for all connectives, e. g. for $\wedge$:
  - $(A \prec B) \supset (C[A \wedge B] \leftrightarrow C[A])$,
  - $(A \leftrightarrow B) \supset (C[A \wedge B] \leftrightarrow C[A])$,
  - $(B \prec A) \supset (C[A \wedge B] \leftrightarrow C[B])$
- simplification of chains: e. g.
  - $((A \prec B) \wedge (B \prec C)) \supset (A \prec C)$
  - $((A \prec B) \wedge (\bigcirc B \prec \bigcirc A)) \supset (\bigcirc A \wedge \bigcirc B)$
  - $((A \prec B) \wedge (B \prec A)) \supset (A \wedge B)$

Prop. Gödel Logic + $\bigcirc$: Axiomatization (3/3)

Difficult ingredient of the proof:

- Technical lemma that constructs **finite** models
  of chains that cannot be "simplified",
  $\prec / \leftrightarrow$ are realized as $< / =$ on [0,1].

Remarks:

- Validity in $\mathcal{L}_{\bigcirc}^{\mathrm{prop}}$ is co-NP.
- Gödel logic + unary operator $\triangle$:

$$\mathfrak{I}(\triangle(A)) := \begin{cases} 1 & \mathfrak{I}(A) = 1 \\ 0 & \mathfrak{I}(A) < 1. \end{cases}$$

Take axioms/rules of [Baaz, 1996].
Propositional Gödel Logic + $\bigcirc$ + $\triangle$ is axiomatizable
by axioms for $\triangle$ and for $\bigcirc$.

First-order Gödel Logic + $\bigcirc$: Not r. e.

Distances can be expressed by $\bigcirc$:

$$\mathfrak{I}((\bigcirc X \supset Y) \vee (\bigcirc Y \supset X)) = \\ = \begin{cases} 1 & \text{if } |\mathfrak{I}(X) - \mathfrak{I}(Y)| \geq r, \\ \max\{\mathfrak{I}(X), \mathfrak{I}(Y)\} & \text{if } |\mathfrak{I}(X) - \mathfrak{I}(Y)| < r. \end{cases}$$

Lemma: Let A be a closed formula in the language of classical logic. Then the following conditions are equivalent:

(a) There is a classical interpretation $\mathfrak{I}'$ such that
$\mathfrak{I}'(A) = 0$ and $|\mathfrak{I}'|$ is finite.

(b) There is $r \in [0,1]$ and a Gödel r-interpretation $\mathfrak{I}$
such that $\mathfrak{I}(t(A)) < 1$, t effective.

Note: $t(A)$ involves binary predicates.

Thus: The set of first-order validities in $\mathcal{L}_{\bigcirc}$ is not r.e.

Generalisation of $\bigcirc$: Definitions (1/2)

Instead of interpreting $\bigcirc$ as $\mathfrak{I}(\bigcirc(A)) = \min\{1, r + \mathfrak{I}(A)\}$, consider a generalisation:

Semantics:
Let $f\colon\ [0,1] \to [0,1]$.
Extend a Gödel interpretation $\mathfrak{I}\colon\ \mathrm{Var} \to [0,1]$
to an **f-Gödel-interpretation** $\mathfrak{I}$ by
$\mathfrak{I}(\bigcirc A) := f(\mathfrak{I}(A))$ for all formulas A.

We will focus on f in special classes.

("strictly monotone")
$$\begin{aligned}\mathcal{F}_{\mathrm{s}} :=\{f\colon [0,1] \to [0,1];\\ &\forall x \in [0,1].\ x \leq f(x)\\ &\forall x, y \in [0,1].\ x < y \Rightarrow (f(x) < f(y) \vee f(y) = 1)\}\end{aligned}$$

Generalisation of $\bigcirc$: Definitions (2/2)

Definitions:

A formula A is $\mathcal{F}_{\mathrm{s}}$-**valid**,
if for all $f \in \mathcal{F}_{\mathrm{s}}$ and for all f-Gödel-interpretations $\mathfrak{I}$,
we have $\mathfrak{I}(A) = 1$.

Generalisation of $\bigcirc$: Axiomatization (1/3)

Theorem:
$\mathsf{IPL} + \mathsf{LIN} + A \supset \bigcirc A + \bigcirc(A \supset B) \leftrightarrow (\bigcirc A \supset \bigcirc B)$
is sound and complete for the $\mathcal{F}_{\mathrm{s}}$-valid formulas.

- … Actually for $(\mathcal{F}_{\mathrm{s}} \cap C([0,1]))$-valid formulas.

- … There is an algorithm to construct a proof for a valid formula.
  Evaluation procedure more involved.
  Easier to construct countermodels for non-reducible chains.

- … Deduction theorem holds (due to $A \supset \bigcirc A$).

Generalisation of $\bigcirc$: Axiomatization (2/3)

Theorem:
$\mathsf{IPL} + \mathsf{LIN} + A \supset \bigcirc A + \bigcirc(A \supset B) \leftrightarrow (\bigcirc A \supset \bigcirc B)$
is sound and complete for the $\mathcal{F}_{\mathrm{s}}$-valid formulas.

Proof idea:

Soundness is a routine matter.

1st method: Completeness by Dummett's result:
Consider a valid formula X that would have no proof.
Use $\bigcirc(A \square B) \leftrightarrow (\bigcirc A \square \bigcirc B)$ for $\square \in \{\supset, \vee, \wedge\}$
to 'shift' rings to variables. Treat $\bigcirc^k A$ as fresh atoms A_k.
'Translate' important ring properties into ring-free formulas T; use as antecedent: $T \supset X$
Use Dummett's result to obtain interpretation I without ring.
T guarantees that $I(A_k) = I(B_n)$ implies $I(A_{k+1}) = I(B_{n+1})$
From that, construct ring interpretation $\mathfrak{I}$ with $\mathfrak{I}(F) < 1$.

Generalisation of $\bigcirc$: Axiomatization (3/3)

2nd method: Completeness

Effective construction of a derivation for a $\mathcal{F}_{\mathrm{s}}$-valid formula:

Proof system:

IPL + LIN + $\bigcirc(A \supset B) \leftrightarrow (\bigcirc A \supset \bigcirc B)$ + $A \supset \bigcirc A$

- case distinction:

$$\frac{(A \prec B) \supset E, \ (A \leftrightarrow B) \supset E, \ (B \prec A) \supset E}{E}$$

- in-depth evaluation for all connectives, e. g.,

  $(A \prec B) \supset (C[A \wedge B] \leftrightarrow C[A])$

  provided that no ring is 'above' the indicated occurence of $A \wedge B$

Future work

- Results only for certain first-order fragments.
- The condition $x \le f(x)$ in $\mathcal{F}_\mathrm{s}$ is not 'natural'.
- Conjecture: The validites w. r. t. the class
  $$\{f : [0,1] \to [0,1];\ f(1) = 1,\ \forall x, y \in [0,1].\ x < y \Rightarrow (f(x) < f(y) \vee f(y) = 1)\}$$
  can be characterized by
  - $\mathrm{IPL} + \mathrm{LIN} + \bigcirc(A \supset B) \leftrightarrow (\bigcirc A \supset \bigcirc B) + \dfrac{A}{\bigcirc A} +$
    $+$ recursively enumerable scheme of axioms
    (However, only a finite number of schemes are needed to prove a valid formula.)

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