

(α, \mathbf{u}) -STATES ON MV-ALGEBRAS

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For every MV-algebra A , a **state** on A is a map $s : A \rightarrow [0, 1]$ which is normalized ($s(\top) = 1$), and additive ($s(x \oplus y) = s(x) + s(y) - s(x \odot y)$)

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Since there are no particular motivations to fix $[0, 1]$ as range of a state (independently on the MV-algebra A where the state is defined), we are going to introduce a **refinement** of the notion of state that have as range a (possibly trivial) ultrapower ${}^*[0, 1]_{\mathfrak{u}}$ of $[0, 1]$, reflecting the (possibly) non-archimedean structure of the MV-algebra A .

MV-ALGEBRAS

An **MV-algebra** is a structure $A = (A, \oplus, \neg, \perp)$ of type $(2, 1, 0)$ such that (A, \oplus, \perp) is a commutative monoid with neutral element \perp , and the following equations hold:

- $\neg(\neg x) = x$
- $x \oplus \top = \top$, where $\top = \neg \perp$
- $x \oplus \neg(x \oplus \neg y) = y \oplus \neg(y \oplus \neg x)$

The class of MV-algebras forms a variety denoted by **MV**.

In any MV-algebra A one can define the following operations:

$$\begin{aligned}x \rightarrow y &= \neg x \oplus y, \quad x \ominus y = \neg(x \rightarrow y), \quad x \odot y = \neg(\neg x \oplus \neg y), \\x \leftrightarrow y &= (x \rightarrow y) \odot (y \rightarrow x), \quad x \vee y = (x \rightarrow y) \rightarrow y, \quad \text{and} \\x \wedge y &= \neg(\neg x \vee \neg y)\end{aligned}$$

In any MV-algebra A one can define an order relation

$$x \leq y \text{ iff } x \rightarrow y = \top.$$

An MV-algebra A is said **linearly ordered** if the order relation is linear.

(1) Every **boolean algebra** is an MV-algebra. Moreover in any MV-algebra A the set of its idempotents element $B(A) = \{x \in A \mid x \oplus x = x\}$ is (the domain of) a boolean algebra, namely $B(A)$ is (the domain of) the largest sub boolean algebra of A .

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(2) The real unit interval $[0, 1]$ with operations \oplus and \neg defined by

$$x \oplus y = \min\{1, x + y\}, \text{ and } \neg x = 1 - x$$

is an MV-algebra denoted by $[0, 1]_{MV}$ and called the **standard MV-algebra**. The variety \mathbb{MV} is generated by $[0, 1]_{MV}$.

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(3) Fix an $n \in \mathbb{N}$. Then the set of all functions from $[0, 1]^n$ into $[0, 1]$ that are continuous, piecewise linear, and such that each piece has integer coefficient, together with the operations \oplus and \neg defined as a pointwise application of those of $[0, 1]_{MV}$ is an MV-algebra (actually the **free MV-algebra over n free generators**).

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(4) **Chang MV-algebra** is an example of an MV-chain that is not semisimple. Any direct product of Chang MV-algebra provides an example of not semisimple MV-algebra.

Let A be an MV-algebra. Then a non-empty subset f of A is said to be a **filter** of A if: (1) $\top \in f$; (2) $x, y \in f$ implies $x \odot y \in f$; (3) $x \in f$, and $y \geq x$, then $y \in f$.

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A filter \mathfrak{p} is said to be a **prime filter** if $\mathfrak{p} \neq A$, and $x \vee y \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. The set of all prime filters of A is denoted by $\text{Spec}(A)$.

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A filter m is said to be an **ultrafilter** if $m \neq A$, and for any filter f , if $f \supseteq m$, then either $f = m$, or $f = A$. We denote by $\mathcal{M}(A)$ the set of all ultrafilter of an MV-algebra A .

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Semisimple MV-algebras provide a strict subclass of MV-algebras.

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Di Nola, Lenzi and Spada ([3]) have recently proved a *uniform* version of Di Nola's theorem.

REMARK

Strictly speaking, Di Nola's theorem is not a "generalization" of Chang-Belluce theorem, i.e. Di Nola's theorem represents *semisimple MV-algebras* as MV-subalgebras of $(*[0, 1])^{\text{Spec}(A)}$, and $\text{Spec}(A) \supset \mathcal{M}(A)$.

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Notice that, if A is semisimple, then its radical $\text{Rad}(A) = \{\top\}$, and hence every proper prime filter (i.e. not maximal) of A is not essential. In other words, when A is semisimple, $\mathcal{E}(A) = \mathcal{M}(A)$.

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LEMMA

For every MV-algebra A , the map $\rho : A \rightarrow \prod_{\mathfrak{e} \in \mathcal{E}(A)} A/\mathfrak{e}$ defined by

$$\rho : a \in A \mapsto (a/\mathfrak{e})_{\mathfrak{e} \in \mathcal{E}(A)},$$

is an embedding. Therefore $\prod_{\mathfrak{e} \in \mathcal{E}(A)} A/\mathfrak{e}$ is a subdirect representation of A .

THEOREM

For every MV-algebra A there exist a cardinal α , and an ultrafilter \mathfrak{u} on α such that:

- A embeds into the algebra of ${}^*[0, 1]_{\mathfrak{u}}$ -valued functions defined over the set of its essential filters $\mathcal{E}(A)$,
- if A is semisimple, then \mathfrak{u} is a principal ultrafilter on α (and hence ${}^*[0, 1]_{\mathfrak{u}} = [0, 1]_{MV}$).

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For every MV-algebra A we say that the pair (α, \mathfrak{u}) that arises from the above theorem, is the **representative pair** of A .

STATES ON MV-ALGEBRAS

Fix any MV-algebra A . A **state** on A is a map $s : A \rightarrow [0, 1]$ satisfying:

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(2): Let $h_1, \dots, h_n : A \rightarrow [0, 1]$ be homomorphisms of A into $[0, 1]_{MV}$, and let $\lambda_1, \dots, \lambda_n$ be positive reals such that $\sum_{i=1}^n \lambda_i = 1$. Then define $s : A \rightarrow [0, 1]$ as: for every $a \in A$,

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(3): Let A be **semisimple**, and let μ be a Borel probability measure on the compact Hausdorff space $\mathcal{M}(A)$ of its maximal filters. Then define, for every $f \in A$,

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Call $\mathcal{S}_{\mathbf{u}}^{\alpha}(A) \subseteq {}^*[0, 1]_{\mathbf{u}}^A$ the set of (α, \mathbf{u}) -states on A . $\mathcal{S}_{\mathbf{u}}^{\alpha}(A)$ is non-empty. In fact:

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(1): Every homomorphism $h : A \rightarrow {}^*[0, 1]_{\mathbf{u}}$ is an (α, \mathbf{u}) -state.

(2): (in particular): for every $\epsilon \in \mathcal{E}(A)$, let η_{ϵ} be the canonical homomorphism of A into the linearly ordered MV-algebra A/ϵ . Then let h be the embedding of A/ϵ into ${}^*[0, 1]_{\mathbf{u}}$. Then

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(3): If A is **semisimple**, then every **(α, \mathbf{u}) -state** s on A , is actually a **state** on A .

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Every (α, u) -state on A is a convex combination (with parameters in ${}^*[0, 1]_u$) of homomorphisms $h_i \in \mathcal{HOM}(A, {}^*[0, 1]_u)$. In other words, for every $s \in \mathcal{S}_u^\alpha(A)$, there are parameters $\lambda_i \in {}^*[0, 1]_u$ such that $\sum_i \lambda_i = 1$, and $s = \sum_i \lambda_i \cdot h_i$.

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$\mathcal{S}_u^\alpha(A)$ is a **convex** subset of ${}^*[0, 1]_u^A$, but Krein Milman theorem cannot be applied in order to show that, $\mathcal{S}_u^\alpha(A)$ coincides with the closure of its extremal points.

Every (α, u) -state on A is a convex combination (with parameters in ${}^*[0, 1]_u$) of homomorphisms $h_i \in \mathcal{HOM}(A, {}^*[0, 1]_u)$. In other words, for every $s \in \mathcal{S}_u^\alpha(A)$, there are parameters $\lambda_i \in {}^*[0, 1]_u$ such that $\sum_i \lambda_i = 1$, and $s = \sum_i \lambda_i \cdot h_i$.

Every homomorphism $h \in \mathcal{HOM}(A, {}^*[0, 1]_u)$ satisfies the following property: h cannot be expressed as $h = \lambda s_1 + (1 - \lambda) s_2$ with s_1, s_2 distinct elements in $\mathcal{S}_u^\alpha(A)$, and $\lambda \in {}^*[0, 1]_u$ satisfying $0 < sh(\lambda) < 1$.

Therefore homomorphisms $h \in \mathcal{HOM}(A, * [0, 1]_{\mathfrak{u}})$ are the **building blocks** for (α, \mathfrak{u}) -states. Not surprisingly there is a close connection between $\mathcal{HOM}(A, * [0, 1]_{\mathfrak{u}})$, and $\mathcal{E}(A)$:

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$$h_{\mathfrak{e}} : a \in A \xrightarrow{\eta_{\mathfrak{e}}} a/\mathfrak{e} \in A/\mathfrak{e} \xrightarrow{\iota} * [0, 1]_{\mathfrak{u}}$$

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Vice versa, for every $h \in \mathcal{HOM}(A, {}^*[0, 1]_{\mathfrak{u}})$,

$$\{x \in A : h(x) = 1\} \in \text{Spec}(A) \supseteq \mathcal{E}(A).$$

COMPARISON WITH STATES

	states	(α, \mathfrak{u}) -states
Range:	$[0, 1]$	${}^*[0, 1]_{\mathfrak{u}} = ([0, 1]^{\alpha})/\mathfrak{u}$
State space:	convex in a locally convex Hausdorff	convex in a locally convex NOT Hausdorff
Extremal points:	$\mathcal{HOM}(A, [0, 1])$	$\mathcal{HOM}(A, {}^*[0, 1]_{\mathfrak{u}})$
Extremal points:	homeomorphic to $\mathcal{M}(A)$	“Close connection” with $\text{Spec}(A)$ and $\mathcal{E}(A)$.

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