

# Leibniz Interpolation Properties

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joint work with

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# Outline

- ▶ Introduction
- ▶ Definitions and characterizations of Leiniz Interpolation Properties
- ▶ Connection with other interpolation properties
- ▶ Final remarks

# Introduction

[H. Kihara and H. Ono, *Interpolation Properties, Beth Definability Properties and Amalgamation Properties for Substructural Logics*, Journal of Logic and Computation, DOI: [10.1093/logcom/exn084](https://doi.org/10.1093/logcom/exn084)]

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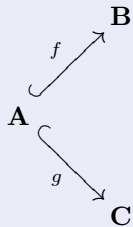
## Definition (Kihara-Ono Interpolation Property)

For every pair of formulas  $\varphi, \psi$ ,

$$\vdash (\varphi \setminus \psi) \wedge (\psi \setminus \varphi) \quad \Rightarrow \quad \exists \delta \text{ s.t. } \begin{cases} \vdash (\varphi \setminus \delta) \wedge (\delta \setminus \varphi) \\ \vdash (\delta \setminus \psi) \wedge (\psi \setminus \delta) \\ \text{var}(\delta) \subseteq \text{var}(\varphi) \cap \text{var}(\psi) \end{cases}$$

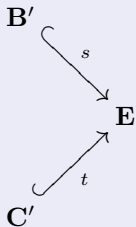
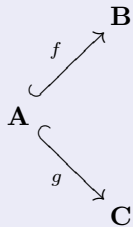
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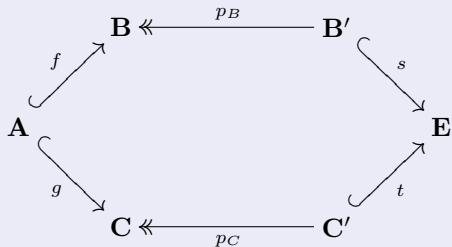
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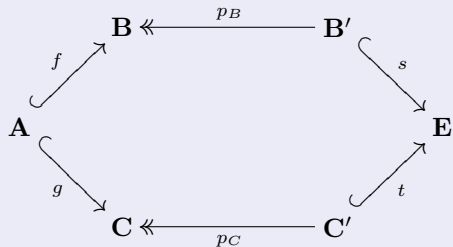
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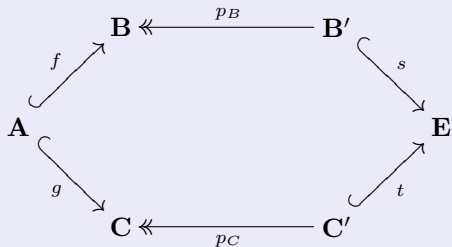
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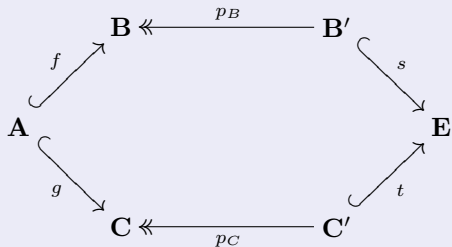
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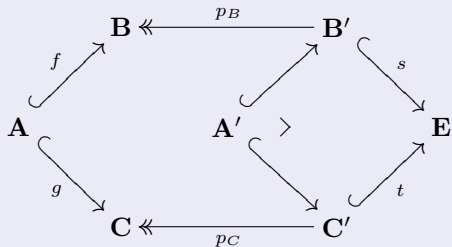
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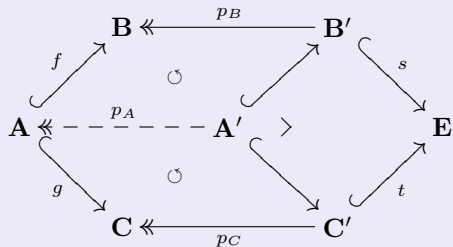
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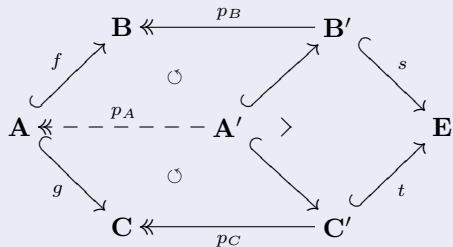
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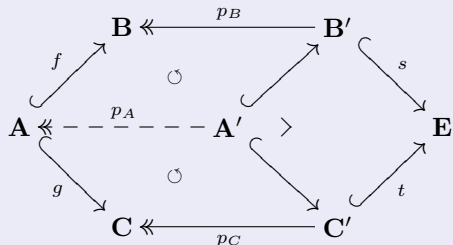


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Feeling a GAP

or

filling a gap!

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- ▶  $\mathcal{S}$  is **equivalential** if there exists a set of equivalence formulas for  $\mathcal{S}$ .

# Looking for a generalization

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# LIP and its characterization

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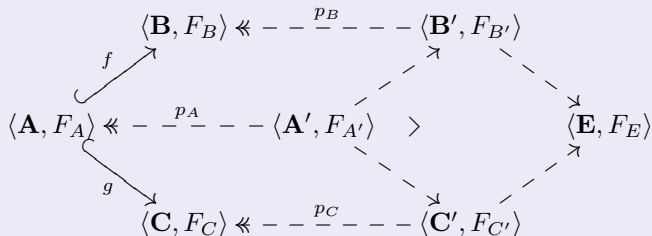
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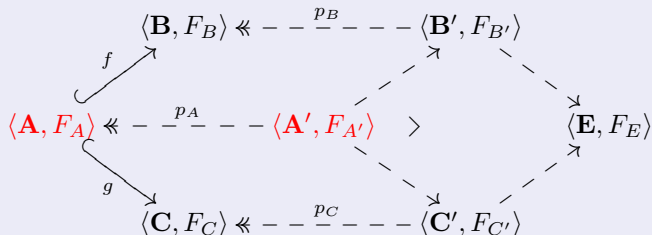
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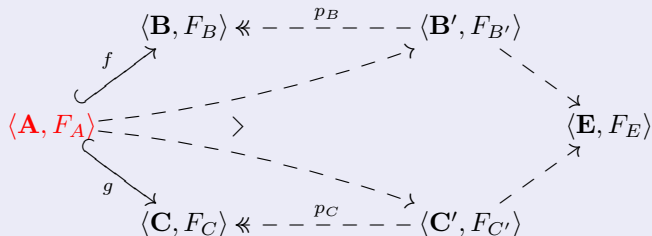
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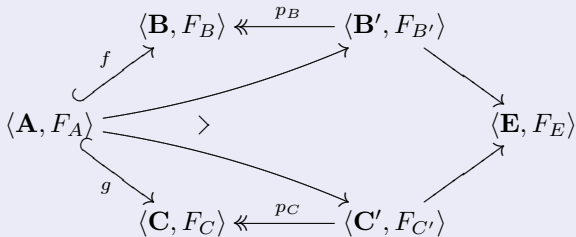


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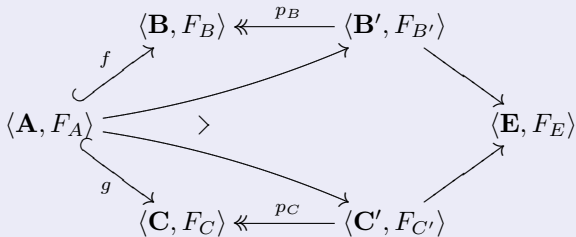


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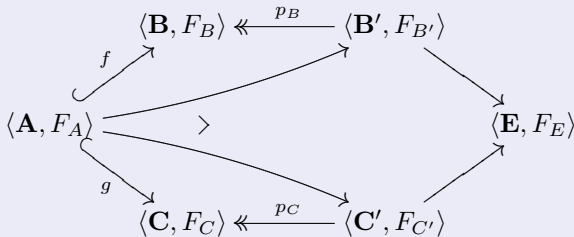


## Asymmetric Leibniz Interpolation Property

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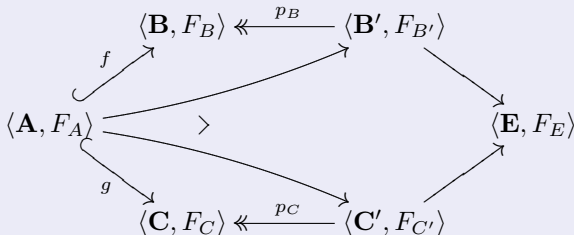


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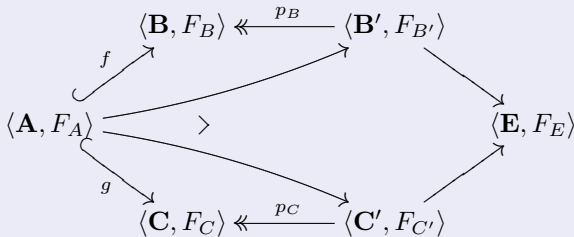
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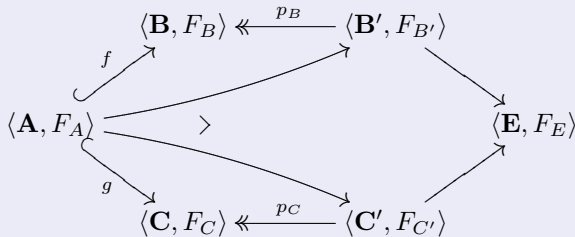
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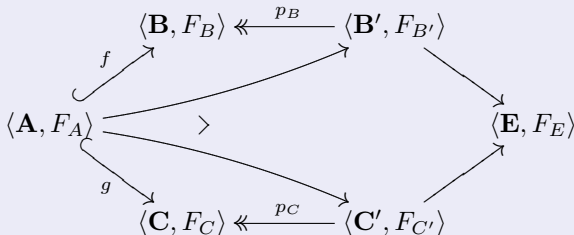
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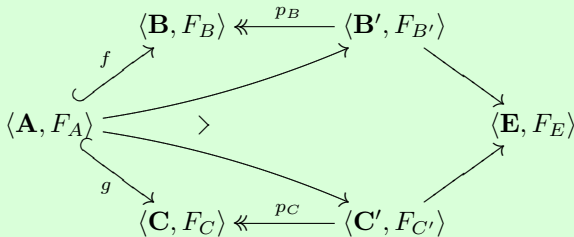
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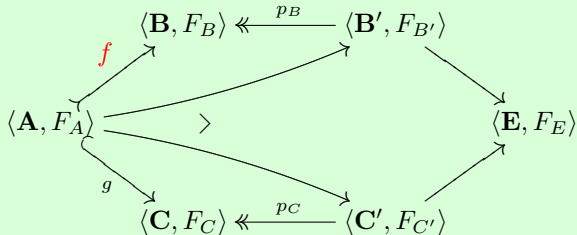
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## d-pullbacks for strict morphisms



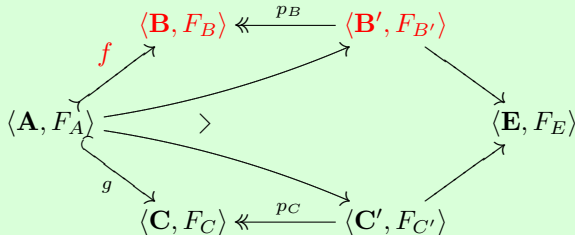
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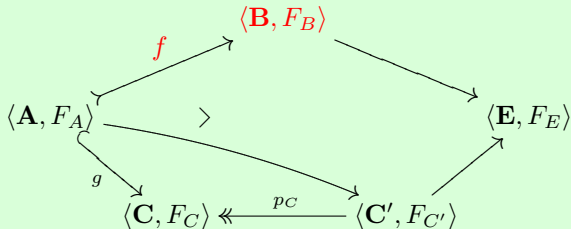
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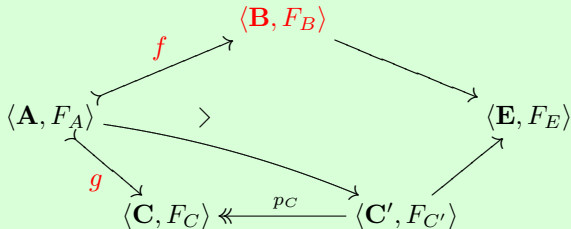
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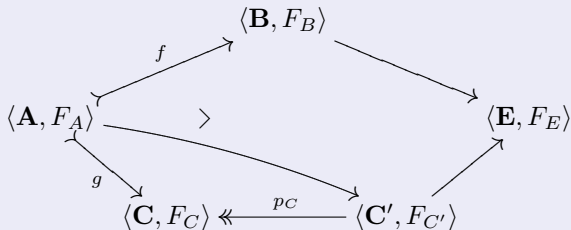
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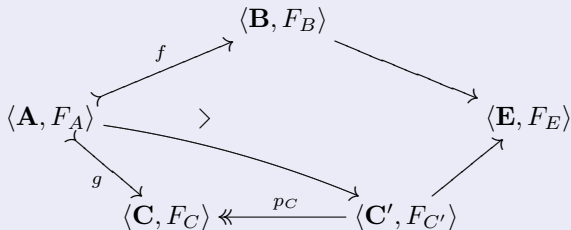
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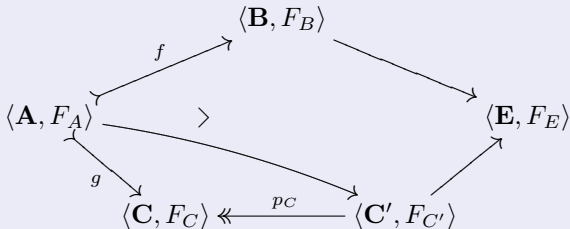
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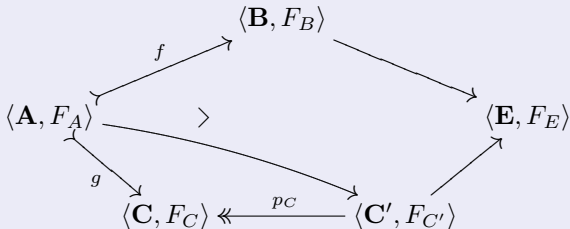
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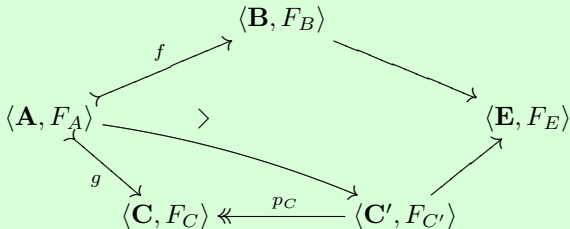
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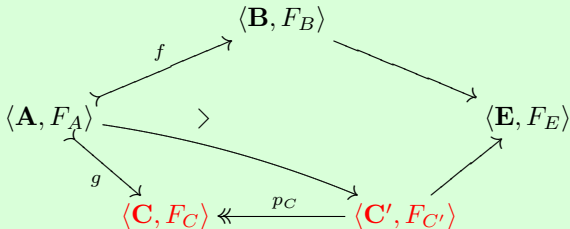
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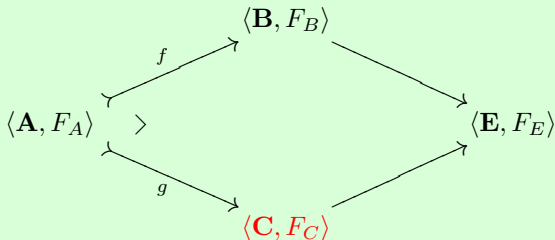
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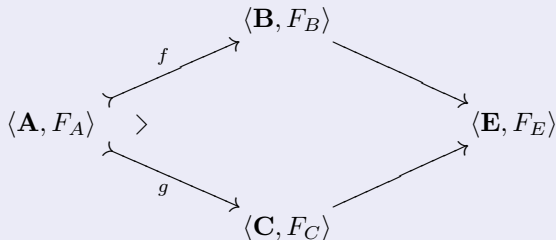
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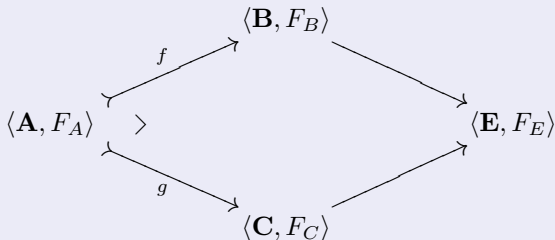
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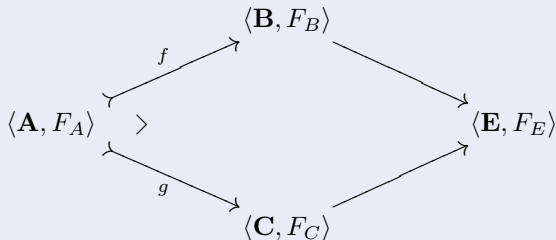
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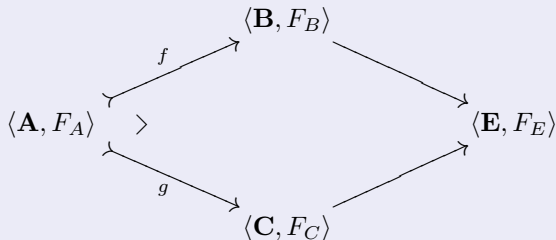
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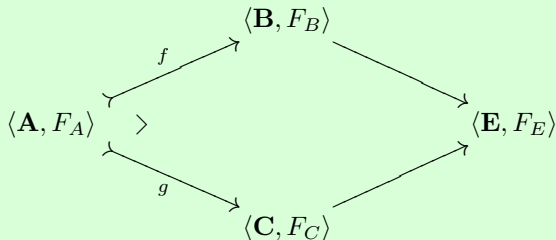
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## Completing pullbacks for strict-arbitrary pairs



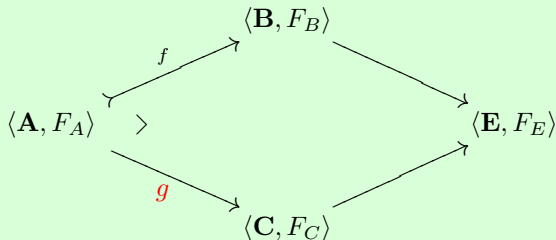
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## Completing pullbacks for strict-arbitrary pairs



## Theorem (Characterization of 'being equivalential')

- ▶ A sentential logic  $S$  is equivalential if and only if for every  $M \in \mathbf{Mod} S$ , the canonical projection  $\pi_M : M \rightarrow M^*$  is a  $\mathbf{Mod}^* S$ -reflection.

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## Corollary

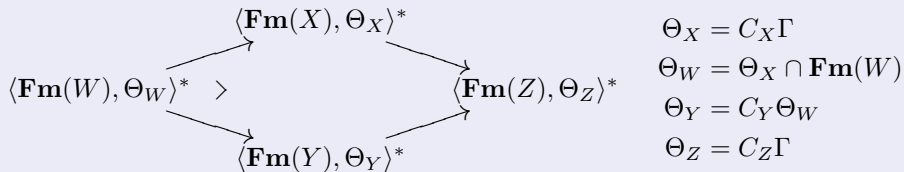
Let  $S$  be an equivalential logic,  $X, Y$  sets, and  $W = X \cap Y$  and  $Z = X \cup Y$ . If  $T_W, T_X, T_Y$  are theories, then the following diagram is a pushout in  $\mathbf{Mod}^* S$ :

$$\begin{array}{ccccc}
 & & \langle \mathbf{Fm}(X), T_X \rangle^* & & \\
 & i \nearrow & & \searrow i' & \\
 \langle \mathbf{Fm}(W), T_W \rangle^* & & & & \langle \mathbf{Fm}(Z), T_Z \rangle^* \\
 & j \searrow & & \nearrow j' & \\
 & & \langle \mathbf{Fm}(Y), T_Y \rangle^* & & 
 \end{array}$$

$$\begin{aligned}
 T_W &\subseteq T_X \cap T_Y \\
 T_Z &= C_Z(T_X \cup T_Y)
 \end{aligned}$$

## Lemma

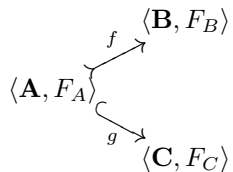
Let  $S$  be an equivalential logic,  $X, Y$  sets,  $W = X \cap Y$  and  $X \cup Y \subseteq Z$ . If  $S$  has the ALIP, then the following diagram is a pullback in  $\mathbf{Mod}^* S$ :



where  $\Gamma \subseteq X$  is arbitrary.

# Proof of the characterization of ALIP

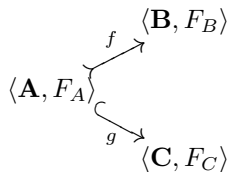
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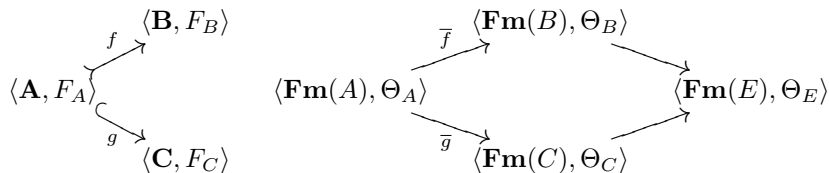
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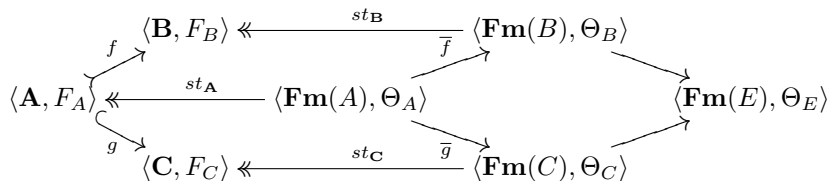
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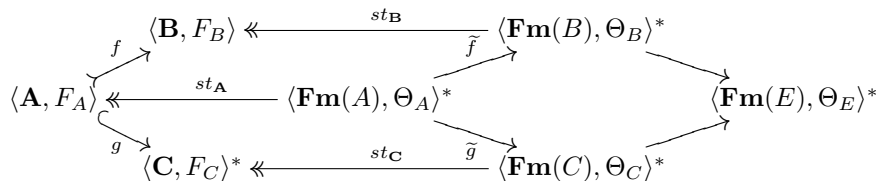
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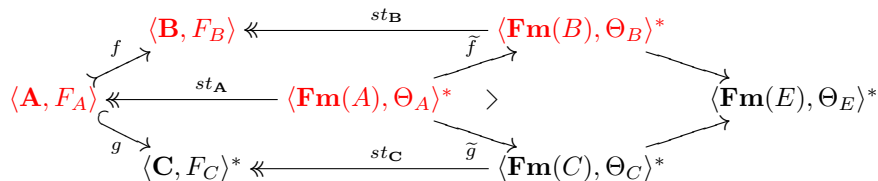
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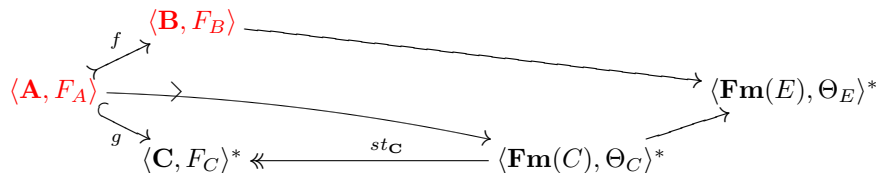
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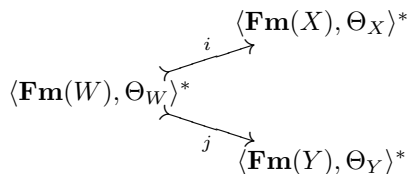
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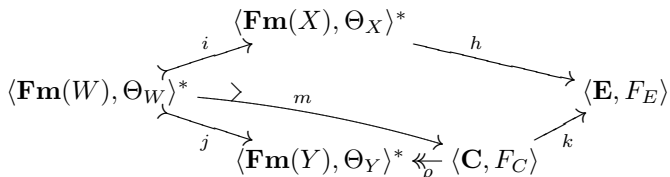
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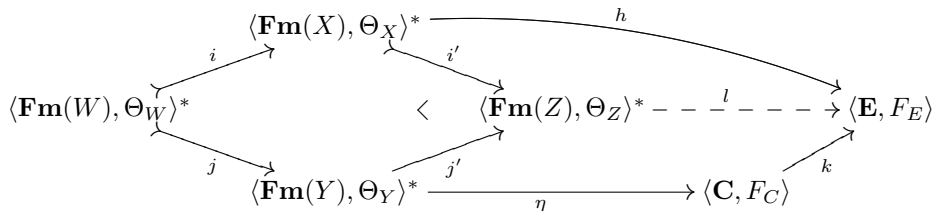
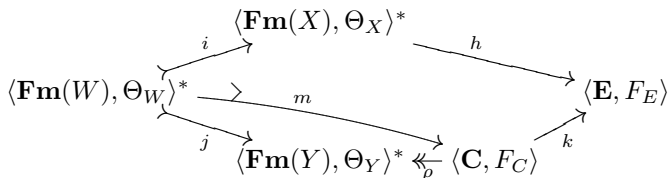
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# A slight modification: LIP'

## Leibniz Interpolation Property'

For every pair of formulas  $\varphi, \psi$ , and every set of formulas  $\Gamma$ ,

if  $\text{var}(\Sigma, \varphi) \cap \text{var}(\Gamma, \psi) \neq \emptyset$  or the language of  $\mathcal{S}$  has a constant, then

$$\langle \varphi, \psi \rangle \in \mathbf{\Omega}(C_{\mathcal{S}}\Gamma) \quad \Rightarrow \quad \exists \delta \text{ s.t. } \begin{cases} \langle \varphi, \delta \rangle \in \mathbf{\Omega}(C_{\mathcal{S}}\Gamma) \\ \langle \delta, \psi \rangle \in \mathbf{\Omega}(C_{\mathcal{S}}\Gamma) \\ \text{var}(\delta) \subseteq \text{var}(\Gamma, \varphi) \cap \text{var}(\Gamma, \psi) \end{cases}$$

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## Theorem

*Every equivalential logic satisfies LIP'.*

# Outline

- ▶ Introduction
- ▶ Definitions and characterizations of Leiniz Interpolation Properties
- ▶ **Connection with other interpolation properties**
- ▶ Final remarks

# Regularity

## Definition

A sentential logic  $\mathcal{S}$  is **regular** if every reduced model  $\langle \mathbf{A}, F \rangle$  of  $\mathcal{S}$  is such that  $F = \{a\}$  for some  $a \in A$ .

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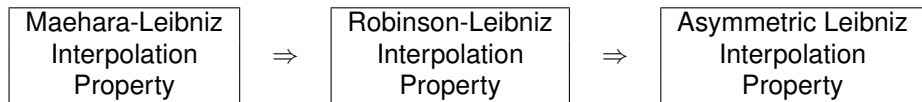
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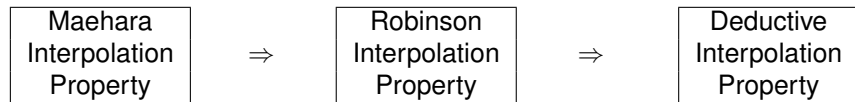
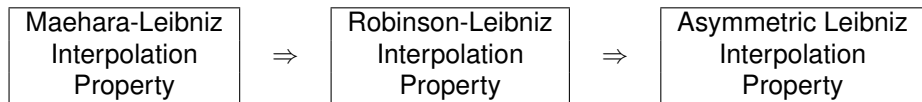
We say that  $\mathcal{S}$  is  **$(\top, \Lambda)$ -regular** if  $\top$  is a constant of the language of  $\mathcal{S}$ ,  $\Lambda$  is a generalized conjunction for  $\mathcal{S}$  and

- (i)  $\vdash_{\mathcal{S}} \top$ ,
- (ii) for every  $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}(X)$ , if  $\Gamma \vdash_{\mathcal{S}} \varphi$  then  $\Lambda(\varphi, \top) \subseteq [\top]_{\Omega(C_{\mathcal{S}}\Gamma)}$ .

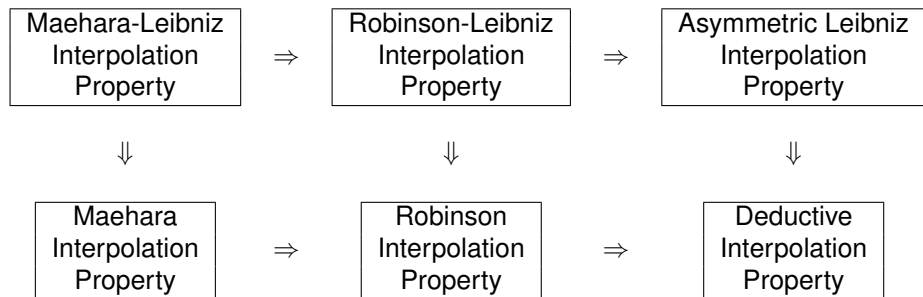
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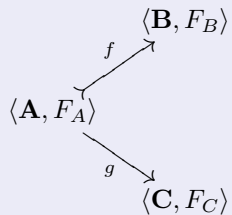
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## Maehara-Leibniz IP

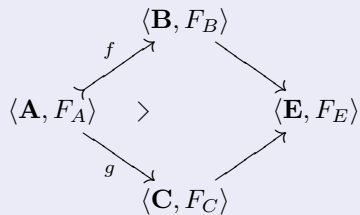
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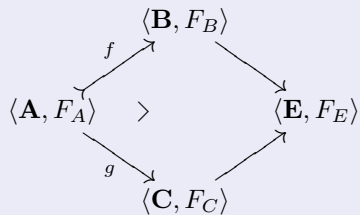
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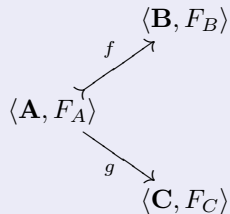


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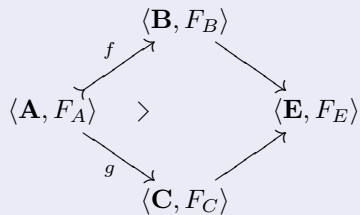
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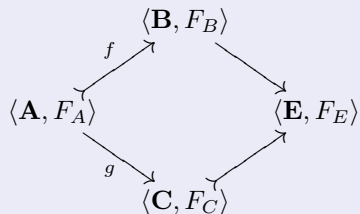
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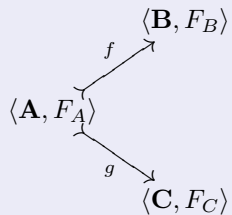
## Maehara IP



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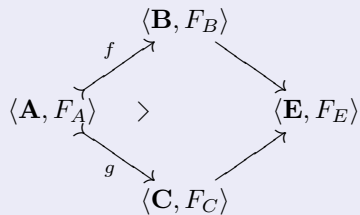
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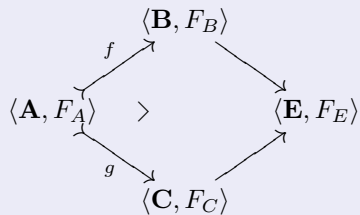
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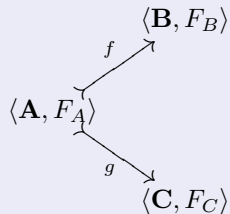


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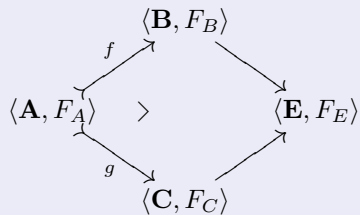
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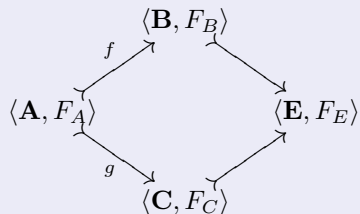
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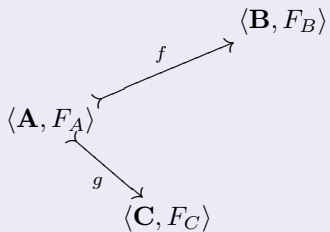
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## Asymmetric Leibniz IP

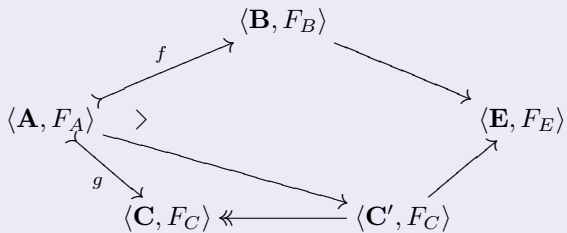
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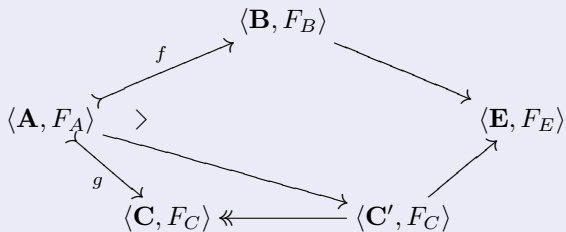
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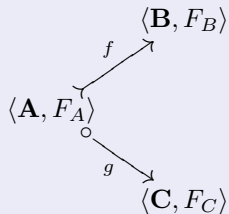


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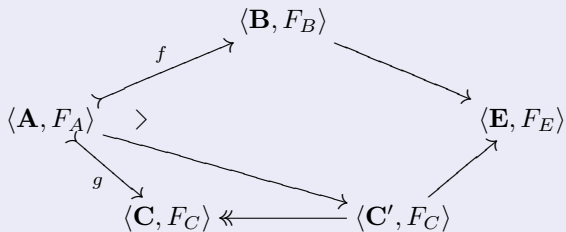
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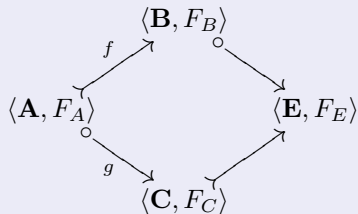
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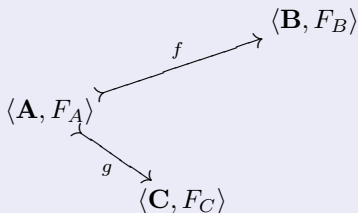
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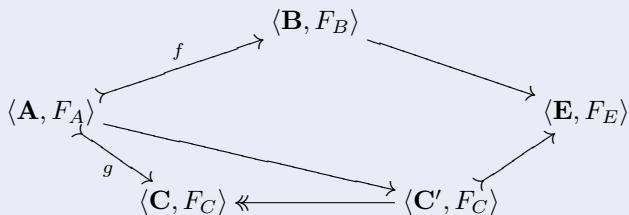


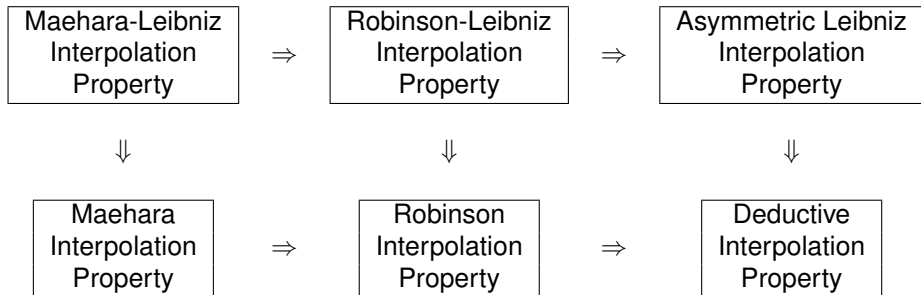
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- ▶  $\perp_\infty$  separates ALIP from Robinson-Leibniz IP
  
- ▶ Beth Definability Properties?
- ▶ Craig Interpolation Properties?
- ▶ Preservation of Intersections by Functors?
- ▶ Examples?

Thank you for your attention!

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- ▶ if  $\pi : [0, 1]^n \rightarrow \pi[0, 1]^m$  is a projection, then  $\pi$  and  $\pi^{-1}$  map rational polyhedra to rational polyhedra.

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Then we prove:

- (1)  $\alpha \vdash_{\perp_{\infty}} \varphi \leftrightarrow \delta$ ;
- (2)  $\beta \vdash_{\perp_{\infty}} \delta \leftrightarrow \psi$ .