Free and Projective Bimodal Symmetric Gödel Algebras

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Introduction

A ”symmetric” formulation of intuitionistic propositional calculus, suggested by various authors (G. Moisil, A. Kuznetsov, C. Rauszer), presupposes that any connective

$$\&, \vee, \rightharpoonup, \top, \bot$$

has its dual

$$\vee, \&, \rightharpoondown, \ \bot, \top,$$

and the duality principle of classical logic is restored.

The notion of double-Browerian algebras was introduced by J. McKinsey and A. Tarski in

`[J. McKinsey and A. Tarski, On closed elements in Closure algebras, Ann. Math, v. 47 (1946), 122-162],`

based on the idea considered by T. Skolem in 1919.

In

`[L. Esakia, The problem of dualism in the intuitionistic logic and Browerian lattices, V International Congress for Logic, Methodology and Philosophy of Sciences, Contributed papers. London, Ontario, Canada, 1975, Section 1, pp. 7-8.]`

double-Browerian algebras were named Skolem algebras.

Heyting-Brouwer logic (alias symmetric Intuitionistic logic Int^2) was introduced by C. Rauszer as a Hilbert calculus with algebraic semantics

`[C. Rauszer, Semi-Boolean algebras and their applications to intuitionistic logic with dual operations, Fund. Math., vol. 83, N. 3 (1974), 219-249.]`

Notice, that the variety of Skolem (Heyting-Brouwerian) algebras are algebraic models for symmetric Intuitionistic logic Int^2.

The well known procedure for embedding the intuitionistic propositional calculus Int into classical modal system $S4$ can be also extended on symmetric intuitionistic logic Int^2 (L. Esakia), which is embedded into bimodal (temporal) logical system $S4^2$ introduced by K. Segerberg

`[K. Segerberg, Modal logic with linear alternative relations,"Theoria", Vol. 38, N 3(1970), 301-322.]`

The language of $S4^2$ consists of $\to, -, H$ ("it always was that", G ("it always will be that"); the temporal connections F ("it will be the case that"), P ("it was the case that") are introduced in the usual way: $F\alpha = -G-\alpha$ and $P\alpha = -H-\alpha$.

Gödel logic G is an extension of intuitionistic logic Int by the linearity axiom

$$(p \rightharpoonup q) \vee (q \rightharpoonup p).$$

Denote by G^2 the extension of symmetric Intuitionistic logic Int^2 by Gödel (the linearity) axiom and dual Gödel axiom.

The well known procedure for embedding the intuitionistic propositional calculus into Gödel-Löb modal system GL (alias, the provability logic) can be also extended on symmetric intuitionistic logic Int^2.

The proof-intuitionistic logic KM

[A. Kuznetsov and A. Muravitsky, On superintuitionistic logics as fragments of Proof Logic extensions, Studia logica, **45** (1986), 77-99]

is the intuitionistic logic Int enriched by $\Box$ as $Prov$ modality satisfying the following conditions:

$$p \rightharpoonup \Box p,\ \Box p \rightharpoonup (q \vee (q \rightharpoonup p)),\ (\Box p \rightharpoonup p) \rightharpoonup p.$$

We investigate the proof-symmetric logic G^2 enriched by two modalities $\Box$ (considered as $Prov$ modality) and $\Diamond$; we denote this logic by MG^2.

We call this logic *a bimodal symmetric Gödel logic*. Semantically the logic MG^2 is defined in the following way:

MG^2 **is the set of all formulas which are valid in all finite Kripke models** (X, R)**, where the binary relation** R **is transitive and irreflexive, while reflexive closure** R^ρ **of** R **is a totally ordered.**

- The description of finitely generated free algebras in the variety $\mathbf{MG^2}$ of algebras corresponding to the bimodal symmetric Gödel logic MG^2, which is equivalent to the description of non-equivalent formulas (with fixed number of variables) in this logic.

- The necessary conditions of finitely generated projective algebras in the variety $\mathbf{MG^2}$.

Let us note that finitely generated free and projective algebras in the variety $\mathbf{G^2}$ (and in the variety $\mathbf{G}$ as well) are finite.

However the finitely generated free and projective algebras in the variety $\mathbf{MG^2}$ are infinite.

Preliminaries

An algebra

$$(T, \vee, \wedge, \rightharpoonup, \rightharpoondown, 0, 1)$$

is a *Skolem algebra* `[Esakia]` (or *Heyting-Browerian algebra*), if

$$(T, \vee, \wedge, 0, 1)$$

- is a bounded distributive lattice,
- $\rightharpoonup$ is an implication (relatively pseudo-complement),
- $\rightharpoondown$ is coimplication (relatively pseudo-difference) on T.

An algebra $(T, \vee, \wedge, \rightharpoonup, \rightharpoondown, 0, 1)$ is said to be G^2-*algebra*, if

(i) $(T, \vee, \wedge, \rightharpoonup, 0, 1)$ is G-algebra, corresponding to Gödel logic;

(ii) $(T, \vee, \wedge, \rightharpoondown, 0, 1)$ is dual G-algebra (alias Browerian algebra with linearity condition: $(p \rightharpoondown q) \wedge (q \rightharpoondown p) = 0$).

G^2-algebras, which are algebraic models of the logical system G^2, represent a proper subclass of Skolem algebras.

MG^2-*algebra* is an algebra $(T, \vee, \wedge, \rightharpoonup, \rightharpoondown, \Box, \Diamond, 0, 1)$, if $(T, \vee, \wedge, \rightharpoonup, \rightharpoondown, 0, 1)$ is G^2-algebra and the operators $\Box, \Diamond$ satisfy the following conditions:

$$p \leq \Box p, \;\; \Box p \leq q \vee (q \rightharpoonup p), \;\; \Box p \rightharpoonup p = p,$$

$$(p \rightharpoonup q) \vee (q \rightharpoonup p) = 1;$$

$$\Diamond p \leq p, \;\; p \rightharpoondown \Diamond p = \Diamond p, \;\; \Diamond(p \vee q) = \Diamond p \vee \Diamond q,$$

$$(q \rightharpoondown p) \wedge (p \rightharpoondown q) = 0.$$

Let us denote the variety (and the category, as well) of all MG^2-algebras by $\mathbf{MG}^2$.

MG^2 is the set of all formulas valid in all finite totally ordered MG^2-algebras.

Let us introduce some abbreviation: $\neg p = p \rightharpoonup 0$, $\ulcorner p = 1 \rightharpoondown p$

A subset $F \subset T$ is said to be a Skolem filter `[Esakia]`, if F is a filter (i. e. $1 \in F$, if $x \in F$ and $x \leq y$, then $y \in F$, if $x, y \in F$, then $x \wedge y \in F$) and if $x \in F$, then $\neg\ulcorner x \in F$.

Proposition 1. *Let* T *be an* MG^2*-algebra. The lattice of all congruences of the algebra* T *is isomorphic to the lattice of all Skolem filters of the algebra* T*.*

A system (X, R), where X is a non-empty set and R transitive relation, is said to be *Kripke model*. We shall say that a subset $Y \subset X$ is an *upper cone* (or cone) if $x \in Y$ and xRy imply $y \in Y$. The concept of a *lower cone* is defined dually. A subset $Y \subset X$ is called *a bicone* if it is an upper cone and a lower cone at the same time.

We say that (X, τ, R) is a perfect Kripke model (or descriptive frame, in another terminology) if

1) (X, τ) is a topological space, which is a Stone space (i. e. Hausdorf, zero-dimensional and compact space),

2) $R^{-1}(x) = \{y : yRx\}$ is closed, for each $x \in X$,

3) the smallest closed set containing a cone is itself a cone,

4) the smallest cone containing a closed set is closed.

Hereinafter instead of (X, τ, R) we will write (X, R). Let (X, R) and (X', R') be perfect Kripke models; a mapping $f : X \to X'$ is said to be strongly isotone (or p-morphism) if

$$f(y)R'x \Leftrightarrow (\exists y')(yRy'\&f(y') = x)$$

for any $x \in X',\ y \in X$.

A perfect Kripke model (X, R) is called *symmetric* if R is order relation, $(X, \widetilde{R})$ is a perfect Kripke model as well, where $x\widetilde{R}y \Leftrightarrow yRx$.

The category of symmetric perfect Kripke models (X, R), where R is an order relation on X, is dually equivalent to the category of Skolem algebras (Heyting-Browerian algebras) `[Esakia]`**.**

A Kripke model (X, R) is called *strongly symmetric* if (X, R_ρ) is a symmetric Kripke model, (X, R_ρ) is a disjoint union of chains, where R_ρ is the reflexive closure of R, and, in addition, for every clopen A of X and every element $x \in A$ there is an element $y \in A - R^{-1}(A)$ such that either xRy or $x \in A - R^{-1}(A)$.

Notice, that if strongly symmetric Kripke model is finite, then R is irreflexive.

Theorem 2. *The category* $\mathbf{MG^2}$ *of* MG^2*-algebras and algebraic homomorphism is dually equivalent to the category* $\mathbf{SK}$ *of strongly symmetric Kripke models and strongly isotone maps.*

Now we illustrate this duality. Let $X \in \mathbf{SK}$ and $A \in \mathbf{MG^2}$. The set $\mathfrak{S}(X)$ of all clopen cones of X is closed under the following operations: the set union, intersection,

$U \rightharpoonup V = -(R^{-1}(U - V) \cup (U - V))$,

$U \rightharpoondown V = R(U - V) \cup (U - V)$,

$\Box U = -R^{-1}(-U)$,

$\Diamond U = R(U)$.

So, the algebra $(\mathfrak{S}(X), \cup, \cap, \rightharpoonup, \rightharpoondown, \emptyset, X)$ is an MG^2-algebra.

Furthermore, for any morphism

$f : (X_1, R_1) \to (X_2, R_2)$ in $\mathbf{SK}$,

$\mathfrak{S}(f) = f^{-1} : \mathfrak{S}(X_2) \to \mathfrak{S}(X_1)$

is a homomorphism.

On the other hand, for each MG^2-algebra A, the set $\mathfrak{M}(A)$ of all prime filters of A with binary relation R on it, defined in the following way:

$xRy \Leftrightarrow (\forall a \in A)(\Box a \in x \Rightarrow a \in y)$,

and topologised by taking the family

$\beta(a) = \{F \in \mathfrak{M}(A) : a \in F\}$,

for $a \in A$, and their complements as a subbase, denoted by $\mathfrak{M}(A)$, is an object of $\mathbf{SK}$; and for each MG^2-algebra homomorphism

$h : A \to B$, $\mathfrak{M}(h) = h^{-1} : \mathfrak{M}(B) \to \mathfrak{M}(A)$

is a morphism of $\mathbf{SK}$.

So, we have two contravariant functors:

$\mathfrak{M} : \mathbf{MG^2} \to \mathbf{SK}$ and $\mathfrak{G} : \mathbf{SK} \to \mathbf{MG^2}$.

These functors establish a dual equivalence between the categories $\mathbf{MG^2}$ and $\mathbf{SK}$.

Proposition 3. *Let T be an MG^2-algebra and (X, R) corresponding to T strongly symmetric Kripke model. Then the lattice of all congruences of the algebra T is anti-isomorphic to the lattice (by the inclusion relation $\subseteq$) of all closed bicones of (X, R).*

There is a one-to-one correspondence between the homomorphic images of a MG^2-algebra T and the closed bicones of corresponding to it strongly symmetric Kripke model $(\kappa X, \kappa R)$, and between subalgebras of MG^2-algebra T and correct partitions of $(\kappa X, \kappa R)$, where *a correct partition* of a perfect Kripke model (Y, R) is an equivalence relation E on Y, such that

- E is a closed equivalence relation, i. e. E-saturation * of any closed subset is closed;

- E-saturation of any bicone is a bicone;

- $(\forall x, y \in Y)(E(x) \cap R^{-1}(E(y)) \neq \emptyset \Rightarrow E(x) \subseteq R^{-1}(E(y))$;
  $(\forall x, y \in Y)(E(x) \cap R(E(y)) \neq \emptyset \Rightarrow E(x) \subseteq R(E(y))$;

- there is a strongly symmetric Kripke model (Z, Q) and a strongly isotone map $f : Y \to Z$ such that $Kerf = E$.

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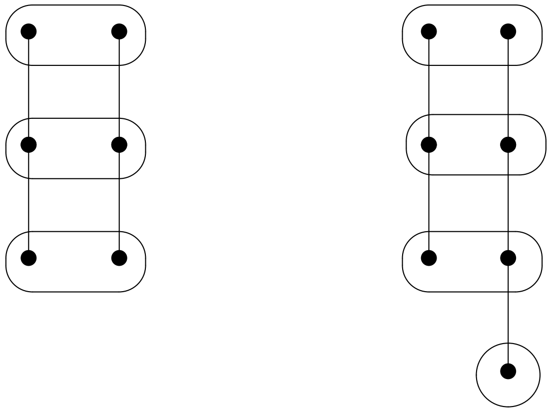
[*] If $V \subseteq Y$ then E-saturation of V is $E(V) = \bigcup_{x \in Y} E(x)$.



(X_1, R_1) (X_2, R_2)

Fig. 1

(X_1, R_1) is a disjoint union of two three-element strongly symmetric Kripke models and (X_2, R_2) is a disjoint union of three- and four-element strongly symmetric Kripke models. The elements in the inside of ovals and circles are equivalent. The partition for (X_1, R_1) is correct, but the partition for (X_2, R_2) is not correct, since $R(a) \cap R(E(b)) \neq \emptyset$ and $E(a) \not\subseteq R(E(b))$, where a is the bottom element of three-element chain and b is the bottom element of four-element chain.

Theorem 4. *If* (X, R) *is a disjoint union of* n*-element strongly symmetric Kripke models* (X_i, R_i) $(i = 1, ..., m)$*, then there exists an order isomorphism* $\varphi_{ij} : X_i \to X_j$ *for every* $i, j \leq m$*. Then the maximal correct proper partition of* (X, R) *will be a partition, for which any class containing* x *contains also the element* $\varphi_{ij}(x)$ *for every* $i, j \leq m$*. If* (X, R) *is a disjoint union of an* n*-element strongly symmetric Kripke model and an* m*-element strongly symmetric Kripke model with* $n \neq m$*, then there does not exist any non-trivial correct partition.*

Theorem 5. *The logic* MG^2 *has finite model property.*

Free Algebras

Let (C_n^m, R_n^m) $(0 \leq m \leq n > 0)$ be a strongly symmetric Kripke model, where C_n^m is an n-element set $\{c_1^m, ..., c_n^m\}$ and R_n^m is an irreflexive and transitive relation such that

$$c_1^m R_n^m c_2^m \ ... \ c_{n-1}^m R_n^m c_n^m .$$

Let $X_n = \coprod_{m=0}^n C_n^m$ be a disjoint union of C_n^m, $R_n = \bigcup_{m=0}^n R_n^m$ and $(X, R) = \bigcup_{n=1}^\infty (X_n, R_n)$. Let g_n^m $(0 \leq m \leq n > 0)$ be an m-element upper cone of C_n^m and $g_n = \{g_n^0, ..., g_n^n\}$. Then $G = \bigcup_{n=1}^\infty g_n \subset X$.

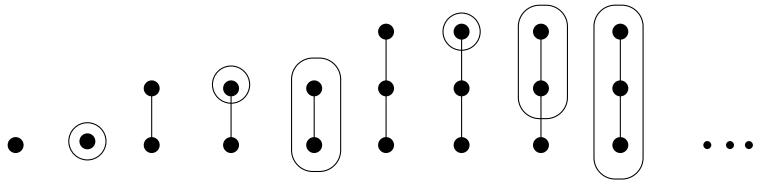


Fig. 2

Let $(T, \cup, \cap, \rightharpoonup, \rightharpoondown, \Box, \Diamond, \emptyset, X)$ be an algebra generated by G by the following operations: the union $\cup$, the intersection $\cap$,

$A \rightharpoonup B = -R_{\rho}^{-1} - (-A \cup B)$,

$A \rightharpoondown B = R_{\rho}(A \cap -B)$,

$\Box(A) = -R^{-1} - (A)$, $\Diamond(A) = R(A)$

for any upper cones of A and B, where R_{ρ} is a reflexive closure of the relation R.

Lemma 6. *The* MG^2*-algebra* $T_n^m = \mathfrak{S}(C_n^m) = (Con(C_n^m), \cup, \cap, \rightharpoonup, \rightharpoondown, \Box, \Diamond, \emptyset, C_n^m)$ *is generated by any element of* T_n^m*, where* $Con(C_n^m)$ *is the set of all upper cones of* (C_n^m, R_n^m)*,* $\cup$ *is the union,* $\cap$ *is the intersection,* $A \rightharpoonup B = -(R_n^m)_\rho^{-1} - (-A \cup B)$*,* $A \rightharpoondown B = (R_n^m)_\rho(A \cap -B)$*,* $\Box A = -(R_n^m)^{-1} - (A)$*,* $\Diamond A = R_n^m(A)$*.*

Theorem 7. *The algebra*

$$(T, \cup, \cap, \rightharpoonup, \rightharpoondown, \Box, \Diamond, \emptyset, X)$$

is a one-generated free MG^2*-algebra with the free generator* G *in the variety* $\mathbf{MG}^2$*.*

Theorem 8. *Any finite cone of* (X, R) *is an element of* T*.*

Projective Algebras

Theorem 9. *Let* A *be* m*-generated subalgebra of the* m*-generated free* MG^2*-algebra* $F_{\mathbf{MG^2}}(m)$ *and* $a_1, ..., a_m$ *the generators of* A*. Let* A_i *be the subalgebra of* A *generated by* a_i *for* $i = 1, ..., m$*. If* $a_i = \Diamond^{n_i} G_i$ *or* $a_i = \Box^{n_i} G_i$ *for some* $n_i \in \omega$*, then the algebra* A *is projective, where* $G_1, ..., G_m$ *are the free generators of* $F_{\mathbf{MG^2}}(m)$*.*

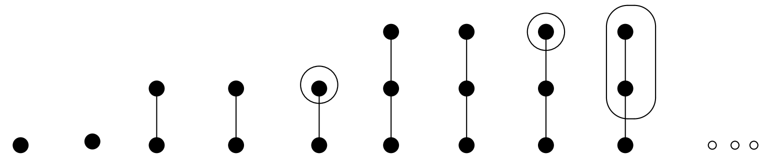


Fig. 3

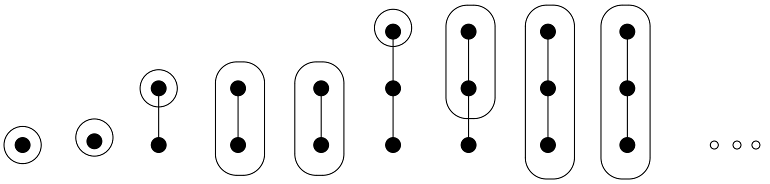


Fig. 4

Let $A_{\Diamond}$ ($A_{\Diamond^n}$) be a one-generated subalgebra of T generated by $\Diamond G$ ($\Diamond^n G$), and $A_{\Box}$ ($A_{\Box^n}$) a one-generated subalgebra of T generated by $\Box G$ ($\Box^n G$).

The element $\Diamond G$ ($\Box G$) is depicted in the Fig. 3 (Fig. 4) by cycles and ovals.

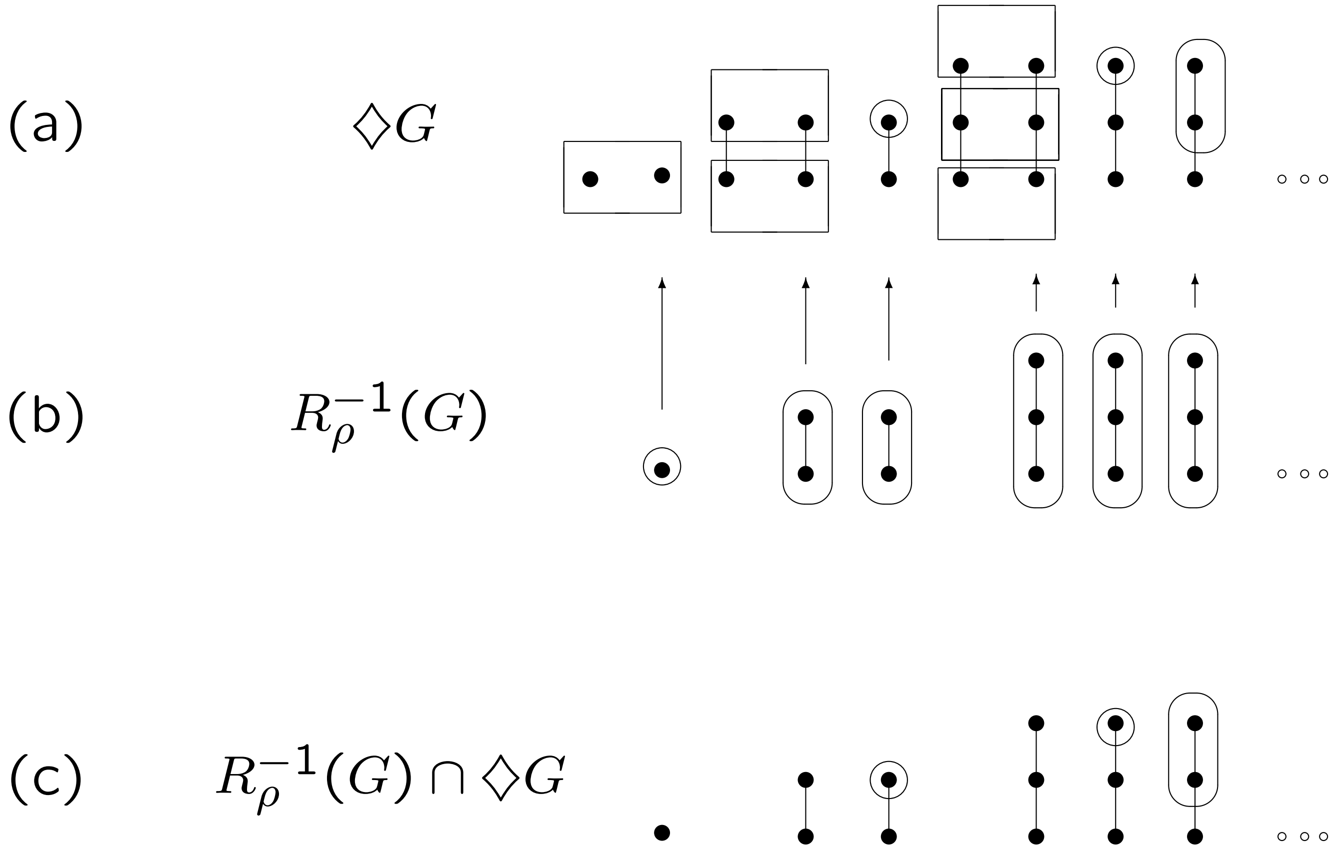


Fig. 5

The result can be generalized on the case when one-generated subalgebra of T is $A_{\Diamond^n}$ ($A_{\Box^n}$).

Theorem 10. *Any finite* MG^2*-algebra is not projective.*

Say that join irreducible element a of MG^2-algebra A has a *height* n if $\Box^n\Diamond^n a = a$. According to this definition C_n^m is a join irreducible element of height n of the algebra T.

Theorem 11. *If A is an m-generated projective MG^2-algebra, then for every positive integer n there exists a join irreducible element $a \in A$ of height n.*

Conjecture. *An m-generated subalgebra A of m-generated free MG^2-algebra is projective if and only if its generators are $\Box^n G_i$ or $\Diamond^n G_i$, where $n \in \omega$ and $i = 1, ..., m$.*

(W, R)

Fig. 6

In the Fig. 6 it is depicted Kripke model (W, R) with the irreflexive and transitive binary relation R such that (W, R_ρ) is the linearly ordered set. Let C be the MG^2-algebra of all upper cones of (W, R). Observe, that C is one-generated. This is the example of infinite finitely generated subdirectly irreducible MG^2-algebra.

Problems

1. Let A be an m-generated MG^2-algebra. Is it true, that if for every positive integer n there exists a join irreducible element $a \in A$ of height n, then A is projective?
2. Does there exist finitely axiomatized subvariety of $\mathbf{MG^2}$ which is not generated by its finite members?
3. Is every subvariety of $\mathbf{MG^2}$ finitely axiomatizable?

Thank You