Some theories over Łukasiewicz logic.

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Recall the joint talk by Cintula and me "First-order fuzzy logics – recent development" on LATD 2008 in Siena. Basic notions (t-norm based fuzzy propositional and predicate fuzzy logics, BL, Ł, G, Π, standard and general semantics etc. assumed to be known. In particular, Łukasiewicz 1920, Łuk.-Tarski 1930; Gödel 1933, H.-Esteva-Godo 1996. These propositional logics have both standard and general completeness, whereas the predicate logics have general completeness w.r.t. corresponding class of general semantics of connectives (class of general linearly ordered algebras (BL-chains, MV-chains, Gödel chains, product chains, ...) BUT from them only Gödel predicate logic has standard completeness.

There are many results on arithmetical hierarchy for these logics; we only mention that the set of standard tautologies of Ł$\forall$ is Π_2-complete whereas this set for $BL\forall$, $\Pi\forall$ (and many others) is non-arithmetical.

Łukasiewicz logic can be defined as the extension of the basic logic by the axiom $\neg\neg\varphi \to \varphi$ of double negation; but if we discuss only this logic we may work with a simpler definition.

Starting connectives are implication and negation; their standard semantics (on [0,1]) is $x \to y = 1$ for $x \leq y$ and $= 1 - x + y$ for $x > y$; $\neg x = 1 - x$. From this one defines $x \& y = \neg(x \to \neg y)$ and other connectives, as $x \wedge y = x \&(x \to y$, $x \vee y = (x \to y) \to y$ and $x \oplus y = \neg x \to y = \min(1, x + y)$.

For any MV-chain the definitions are analogous; recall that MV-chains are representable as intervals $[0, e]_G$ where G is a linearly ordered Abelian group, e is a positive element of G and the operations are defined analogously to the standard case.

The predicate calculus Ł$\forall$ results by adding to the deductive system of Ł the axioms
$(\forall x)A(x) \to A(t)$ for t free for x in A
$(\forall x)(B \to A) \to (B \to (\forall x)A$ if x is not free in B
and the deduction rule of generalization.

Fuzzy logic and arithmetic.

Restall (a paper from 1993) studies a version of Peano arithmetic in Łukasiewicz predicate logic with its standard semantics. The language has equality, constant $\bar{0}$,operations of successor, addition and multiplication, natural axioms for $=$,
$x = y \to x' = y'$, $\bar{0} \neq x'$,
$x + \bar{0} = x$, $x + y' = (x + y)'$,
$x\bar{0} = \bar{0}$, $xy' = xy + x$
and induction rule
$A(\bar{0}), (\forall x)(A(x) \to A(x')) \models (\forall x)A(x)$.
He calls this theory $Ł_\infty$ and shows that it proves the same formulas a (classical) Peano arithmetic. Then he adds a truth predicate T and the “Tarski scheme” $T(\bar{A}) \equiv A$ (for all sentences A). This theory is called $Ł_\infty^{\#}$.

As usual, he calls (arithmetical) theory X ω-inconsistent if there is a formula $A(x)$ such that for each natural n $X \models A(\bar{n})$ but also $X \models \neg(\forall x)A(x)$. And he proves

Theorem. The theory Ł$^{\#}_{\infty}$ is ω-inconsistent. The proof is not too complicated. Note his use of $\models$, not $\vdash$, i.e. semantic proof, he uses only models with the truth set [0,1] (standard semantics) and this semantic consequence is not recursively axiomatizable.

The theory Ł$_{\infty}^{\#}$ is also investigated in he paper Hájek-Paris-Shepherdson from 2000 (without knowledge of Restall's paper). They call the theory PAŁTr and prove
(1) the standard model N of arithmetic has no extension to a model of PAŁTr, (2) But the theory is consistent and has a non-standard model, (3) BUT adding the axiom schema that the truth predicate commutes with connectives makes the theory inconsistent.

Yatabe (A note on Hájek-Paris-Shepherdson theorem 2005) has a simplified proof of (1) and (2).

What about very weak arithmetic over fuzzy logic; can we still have something like Gödel's incompleteness theorem? Recall Robinson's theory Q and investigate it over Ł∀.

The theory FQ^- over Ł∀ (with crisp equality) has the following language: the unary function symbol S, the constant constant $\bar{0}$ and ternary predicates A, M s (addition, multiplication). The axioms are:

(Q1) $S(x) \neq \bar{0}$
(Q2) $S(x) = S(y) \to x = y$
(Q3) $x \neq 0 \to (\exists y)(x = S(y))$
(Q4) $A(x, 0, y) \equiv x = y$
(Q5) $A(x, S(y), z) \equiv (\exists u)(A(x, y, u) \mathbin{\&} z = S(u))$
(Q6) $M(x, 0, y) \equiv y = \bar{0}$
(Q7) $M(x, S(y), z) \equiv (\exists u)(M(x, y, u) \mathbin{\&} A(u, x, z))$
(Q8) $x \leq y \equiv (\exists z)A(z, x, y)$.

Numerals are defined as usual: $\bar{m} = \underbrace{S \ldots S(\bar{0})}_{m}$

In my paper from 2007 I prove that this theory is essentially incomplete (thus Gödel's 1. incompleteness).

And in another paper from 2010 I prove that the theory is essentially undecidable (no consistent axiomatizable extension is decidable).

Caution: If we prove essential incompleteness of a theory over BL∀ we cannot imitate the usual proof of essential undecidability due to the weaker deduction theorem: If $T, \varphi \vdash \psi$ then for some natural n, $T \vdash \varphi^n \to \psi$. (The same for other fuzzy logic; only Gödel logic (with idempotent conjunction) has the "classical" deduction theorem. And I have shown that there is a decidable T over Ł∀ and a sentence φ such that $T \cup \{\varphi\}$ is not decidable.

(Mathematical) fuzzy set theory?

(At least) two ways: First, fuzzy set theory analogous to Zermelo-Fraenkel set theory. Papers by Zuzana Haniková (some with me) - the universum as the limit (union) of iterated power building. Well analogous to the classical ZF.

Recall Russel's paradox in classical logic: some formulas do not determine sets.
If $R = \{x \,|\, x \notin x\}$ then $[R \in R$ iff $R \notin R]$
- contradiction.

Comprehension: there is a set $\{x \,|\, \varphi(x, \ldots)\}$ − for which φ is this assumed (provable, true)? Can we have in a fuzzy set theory more comprehension??

Skolem: Mengenlehre gegründet auf einer Logik mit undendlich vielen Wahrheitswerten 1957. His logic is Łukasiewicz; he constructs a model in which the comprehension holds true for all open (quantifierfree) φ. Nice and not too complicated construction.

Other contributions by Chang, Fenstad and others; most important White 1979: The consistency of the axiom of comprehension in the infinite-valued logic of Łukasiewicz. Some papers by Chang (1963), Fenstad 1964.

Most important: R.B.White: The consistency of the axiom of comprehension in the infinitely valued logic of Łukasiewicz Journ. of Phil. Logic 8 (1979) 502-537

Cantor-Łukasiewicz set theory $C\text{Ł}_0$ (consistency was proved by White)– a set theory over Łukasiewicz predicate logic with the binary predicate $\in$ whose terms are object variables and expressions $\{x \,|\, \varphi(x, \ldots)\}$ for each formula $\varphi(x, \ldots)$; atomic formulas have the form $t \in s$ where t, s are terms.
Full comprehension axiom schema (for each formula $\varphi(x, \ldots)$ (not containing u freely), $u \in \{x \,|\, \varphi(x, \ldots)\} \equiv \varphi(u)$ is an axiom).
Two equality predicates: Leibniz equality $x = y \equiv (\forall z)(x \in z \equiv y \in z)$ and the extensional equality $x =_e y \equiv (\forall z)(z \in x \equiv z \in y)$.
The former equality is crisp: $C\text{Ł}_0 \vdash x = y \vee x \neq y$, whereas the latter is not; crispness of $=_e$ implies in $C\text{Ł}_0$ crispness of $x \in y$, which with full comprehension gives Russel's paradox and inconsistency.

We apply the results to the set ω of natural numbers (to be defined) and then we develop some arithmetic (which is difficult since we do not assume ω to be a crisp set not knowing whether such assumption is consistent); finally we prove that CŁ$_0$ is essentially undecidable.

Recall: Łukasiewicz predicate fuzzy logic Ł$\forall$ - recall two semantics - standard (models are interpretations of a predicate language over the real unit interval as the set of truth values endowed with Łukasiewicz truth functions of implication, negation and (definable) truth function of strong conjunction and truth constants 0, 1) and general semantics where models are interpretations of a predicate language over and MV-chain (linearly ordered MV-algebra). With the standard semantics the logic is not recursively axiomatizable, whereas with the general semantics it is. Here we use the latter and deal with provability in CŁ$_0$.

Natural numbers

Let $S(x)$ denote the singleton of x, i.e. $S(x) = \{y|y = x\}$. (Another usual notation is {x}.) The set of natural numbers is defined as

$$\omega = \{x|x = \emptyset \vee (\exists y)(y \in \omega \wedge x = S(y))\}.$$

Note that $\wedge$ can be equivalently replaced by & due to the crispness of $=$. Such thing will be repeatedly important in the sequel.

Remark. There are infinitely many pairwise different (not $=$-equal) sets, each of them $=_e$-equal to ω.

Note that $C\text{Ł}_0$ with its standard semantics (once more: models over the standard Łukasiewicz algebra $[0,1]_{\text{Ł}}$ of truth functions) was studied by Yatabe; among other things he proves that this theory (called $\mathbf{H}$ by him) is ω-inconsistent, i.e. there is a formula (say, $\alpha(x)$) such that for each model $\mathbf{M}$ with standard semantics the formula $\alpha(\bar{n})$ is true for each natural n but the formula $(\exists x)(x \in \omega \,\&\, \neg\alpha(x))$ is also true in $\mathbf{M}$. The proof is rather tricky and does not use any notion of addition or multiplication.

We discussed a very weak fuzzy arithmetic (denoted Q^-) which is essentially undecidable. We cannot try to interpret Q^- in CŁ$_0$ by restricting quantifiers to ω since we do not assume ω to be crisp; but we shall try to develop the Q^--like arithmetic inside CŁ$_0$ as far as possible. We define the predicates of addition and multiplication and first prove some axioms on them as similar to those of Q^- as possible. Then we shall prove essential undecidability of CŁ$_0$. Define *numerals*: $\bar{0} = \emptyset$, for natural $n > 0$, let $\bar{n}$ be $S(S(\dots S(x)\dots))$ (n copies of S).

Lemma. CŁ$_0$ proves $x \in \omega \equiv S(x) \in \omega$. It also proves
($Q1$) $S(x) \neq \bar{0}$,
($Q2$) $S(x) = S(y) \to x = y$.

Proof easy; for (Q2): $S(x) = S(y)$ implies $S(x) =_e S(y)$ implies $(\forall u)(u = x \equiv u = y)$ implies $x = y$.

Definition. $A = \{(x, y, z) | x \in \omega \wedge y \in \omega \wedge z \in \omega \wedge$ $[(y = \bar{0} \wedge z = x) \vee (\exists u, v)(y = S(u) \wedge z = S(v) \wedge$ $(x, u, v) \in A)]\}$;

$M = \{(x, y, z) | x \in \omega \wedge y \in \omega \wedge z \in \omega \wedge$
$[(y = \bar{0} \wedge z = \bar{0}) \vee (\exists u, v)(y = S(u) \wedge$
$\wedge ((x, u, v) \in M \,\&\, (v, x, z) \in A))]\}$;

$x \leq y \equiv$
$\equiv (x \in \omega \wedge y \in \omega \wedge (\exists z)(z \in \omega \wedge (z, x, y) \in A))$.

Recall that these definitions are correct by the (Cantini's) fixed point theorem, as mentioned above. Observe that all $\wedge$ symbols in [...] can be equivalently replaced by & due to the crispness of $=$; but take care of the & in the definition of M; this in not to be replaced by $\wedge$. Also the $\wedge$ is impossible to replace equivalently by & in $x \in \omega \wedge y \in \omega \wedge z \in \omega$. And we shall need $\wedge$ there; one example is already in the next lemma.

Let us write $A(x, y, z)$ instead of $(x, y, z) \in A$ and similarly for M.

Lemma. $C\text{Ł}_0$ proves the following:

(Q3) $x \in \omega \to [x = \bar{0} \vee (\exists y)(x = S(y))]$

(Q4) $A(x, \bar{0}, z) \equiv (x \in \omega \wedge z = x)$

(Q5) $A(x, S(y), z) \equiv (\exists v)(A(x, y, v) \wedge z = S(v))$

(Q6) $M(x, \bar{0}, z) \equiv (z = \bar{0} \wedge x \in \omega)$

(Q7) $M(x, S(y), z) \equiv (\exists v)(M(x, y, v) \,\&\, A(v, x, z))$

(Q8) $x \leq y \equiv (\exists z)A(z, x, y)$

Proofs easy.

Theorem.

For each $m, n \in N$,

(1) $C\text{Ł}_0 \vdash A(\bar{m}, \bar{n}, x) \equiv x = \overline{m + n}$.

(2) $C\text{Ł}_0 \vdash M(\bar{m}, \bar{n}, x) \equiv x = \overline{m \cdot n}$.

(3) $A(\bar{m}, \bar{n}, x), M(\bar{m}, \bar{n}, x)$ is crisp in $C\text{Ł}_0$.

Theorem. $C\text{Ł}_0$ proves

(1) $A(x, y, \bar{0}) \to (x = \bar{0} \,\&\, y = \bar{0})$

(2) $M(x, y, \bar{0}) \to (x = \bar{0} \vee y = \bar{0})$

(3) $A(x, \bar{1}, S(x))$

(4) $x \in \omega \to (\bar{0} \leq x)$

(5) $S(x) \leq \overline{n+1} \to x \leq \bar{n}$

(6) $A(S(x), \bar{n}, z) \equiv A(x, \overline{n+1}, z)$

(7) $\bar{n} \leq x \to (x = \bar{n} \vee \overline{n+1} \leq x)$

(8) $\vdash \bar{m} \neq \bar{n}$ for $m \neq n$

(9) $\vdash x \leq \bar{n} \equiv (x = \bar{0} \vee x = \bar{1} \vee \ldots \vee x = \bar{n})$

(10) $\vdash x \in \omega \to (x \leq \bar{n} \vee \bar{n} \leq x)$

Fact. In each model of CŁ$_0$ the (fuzzy) set ω contains a crisp initial segment isomorphic to the standard model $\mathbf{N}$ of natural numbers. (Caution: this segment is not claimed to be a set of the model!)
In $\mathbf{N}$ we define the interpretation of the predicates A, M in the obvious way. Now we define: Σ_0^--formulas are formulas built from 0, $=$, S, A, M using connectives $\wedge, \vee, \neg$ and bounded quantifiers. Σ_1^--formulas are formulas of the form $(\exists x)\varphi$ with φ being Σ_0^-.

Observe: (1) Over **N**, each Σ_1-formula is equivalent to a Σ_1^--formula.
(2) For each closed Σ_0^--formula φ there is a closed quantifier-free formula $\varphi^{\#}$ such that the equivalence $\varphi \equiv \varphi^{\#}$ is true in $\mathbf{N}$ and also provable in CŁ$_0$.
(3) Each closed Σ_0^- formula true in $\mathbf{N}$ is provable in CŁ$_0$. (An open-closed formula true in $\mathbf{N}$ is true in each model of CŁ$_0$ by the theorem above.)

Theorem. (cf. H-Pudlák III.1.25) Let $F : N \to N$ (a possibly partial function) be Σ_1 defined. Then there is a $\Sigma_1^-(\omega)$-formula $\varphi(x, y)$ such that, for each $m \in dom(F)$,

$$C\text{Ł}_0 \vdash \varphi(\bar{m}, y) \equiv y = \overline{F(m)}.$$

One can check that the proof of essential undecidability of the theory $FQ^{\sim}$ s can be used to get a proof of essential undecidability of our theory $C\text{Ł}_0$.

Definition (1) A set $X \subseteq N$ is definable in a theory T (over Ł∀) extending CŁ$_0$ by a formula $\Psi(x)$ if for each $n \in N$, $n \in X$ implies $T \vdash \Psi(\bar{n})$ and $n \notin X$ implies $T \vdash \neg\Psi(\bar{n})$.

(2) A function $F : N \to N$ is definable in T by a formula $\Phi(u, v)$ if for each $n \in N$, $T \vdash \Phi(\bar{n}, v) \equiv v = \overline{F(n)}$.

By our preceding theorem every recursive function (of one argument) is definable in CŁ$_0$ and in each its extension. In the sequel we identify formulas with their Gödel numbers. $\varphi_n(x)$ is the n-th formula with at most one free variable in the natural ordering.

Lemma. Let T be a theory over Ł∀ extending CŁ$_0$. Let D be the function satisfying $\varphi_n(\bar{n}) = \varphi_{D(n)}$ for each n, let V be the set of all formulas provable in T. If both D and V are definable then T is contradictory. (Cf. Theorem 1 of Tarski-Mostowski-Robinson.)

Theorem. The theory CŁ$_0$ is essentially undecidable, i.e. each its consistent recursively axiomatized extension is undecidable. (This is a corollary of the preceding lemma.)

Theorem. The theory CŁ$_0$ is essentially incomplete, i.e. each its consistent recursively axiomatized extension is incomplete. (Recall that a theory over Ł∀ is called complete if for each pair φ, ψ of sentences at least one of the formulas $\varphi \rightarrow \psi$ and $\psi \rightarrow \varphi$ is provable.)

In his paper "The recursion contradicts to the induction within Łukasiewicz predicate logic" (accepted 2007 in Math. valued logic and cognition) Yatabe presents various interesting results for $C{\L}_o$ with general semantics. He does not prove ω-inconsistency it is not difficult to use his results to get a proof of ω-inconsistency of $C{\L}_o$ with general semantics, equivalently, of deductive ω-inconsistency of this axiomatizable) theory.

Some authors, including Yatabe proved before (semantical) ω-inconsistency of $C{\L}_o$ with standard semantics; but the proof of ω-inconsistency for general semantics (thus deductive ω-inconsistency) is new.