On logics with truth constants for delimiting idempotents

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General Framework

Propositional Basic Fuzzy Logic

Standard semantics given by continuous t-norms on $[0,1]$

Decomposition into Ł, G, and Π-components

Objective: Given a standard BL-algebra $[0,1]_*$, introduce truth constants for the delimiting idempotents of the components and express the structure of $[0,1]_*$ using the truth constants[1]

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[1] Cf. F. Esteva, J. Gispert, L. Godo, C. Noguera, *Adding truth-constants to logics of continuous t-norms: Axiomatization and completeness results*, FSS 158 (2007).

Propositional BL—Deductive system

- (A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
- (A2) $(\varphi \& \psi) \to \varphi$
- (A3) $(\varphi \& \psi) \to (\psi \& \varphi)$
- (A4) $(\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi))$
- (A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$
- (A5b) $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
- (A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
- (A7) $0 \to \varphi$

The deduction rule of BL is modus ponens.

Propositional BL—Semantics and Decomposition Theorem

BL is the logic of continuous t-norms;
i.e., it is complete w. r. t. propositional 1-tautologies of all standard algebras.

Theorem

Let $*$ be a continuous t-norm. Denote I the closed set of idempotents of $*$, $\mathcal{I}_{\mathrm{o}}$ the open intervals whose union is the complement of I, and assume $[a, b] \in \mathcal{I}$ iff $(a, b) \in \mathcal{I}_{\mathrm{o}}$. Then
(i) *For each $[a, b] \in \mathcal{I}$, $*$ on $[a, b]$ is isomorphic either to the product t-norm (on $[0, 1]$) or to Łukasiewicz t-norm (on $[0, 1]$).*
(ii) *If for $x, y \in [0, 1]$ there is no $[a, b] \in \mathcal{I}$ such that $x, y \in [a, b]$, then $x * y = \min(x, y)$.*

Note that analogous result holds for saturated BL-chains.

Expanding the language

Let $[0,1]_*$ be a standard BL-algebra given by $*$.
Denote $\mathrm{EP}(*)$ the idempotent elements delimiting the Ł-, G-, and Π-components of $*$ ("endpoints").

For any $*$, the set $\mathrm{EP}(*)$ is countable.
Moreover, we may assume $\mathrm{EP}(*) \subset \mathbb{Q}$.

Given $*$ with rational endpoints $\mathrm{EP}(*)$, introduce a set of truth constants

$$\mathcal{C}_* = \{c_r : r \in \mathrm{EP}(*), r \neq 0, 1\}$$

The canonical semantics is $e(c_r) = r$ for any evaluation e in $[0,1]_*$.

Successor function

Fix $*$ with a rational set of endpoints $\mathrm{EP}(*)$. For $r \in \mathrm{EP}(*)$, denote $r^+ = \min\{x : x \in \mathrm{EP} \, \mathrm{and} \, r < x\}$; set $r^+ = r$ if no such x exists.

We can define the $+$ function on $\mathcal{C}_*$, by putting $c_r^+ = c_{(r^+)}$.

The logic $\mathrm{BL}_{\mathrm{EP}(*)}$

Definition

Let $*$ be a continuous t-norm with a rational set of endpoints EP. The axioms of the logic $\mathrm{BL}_{\mathrm{EP}(*)}$ are the axioms of BL plus the following formulas:

$$\begin{array}{lr}
(\mathrm{EP}_1^r) & c_r \,\&\, c_r \equiv c_r \text{ for each } r \in \mathrm{EP} \\
(\mathrm{EP}_2^{r,r'}) & c_r \to c_{r'} \text{ for each } r, r' \in \mathrm{EP} \text{ s.t. } r < r' \\
(\mathrm{EP}_3^{r,r'}) & (c_{r'} \to c_r) \to c_r \text{ for each } r, r' \in \mathrm{EP} \text{ s.t. } r < r'
\end{array}$$

The deduction rule is modus ponens.

$\mathrm{BL}_{\mathrm{EP}(*)}$-algebras

Definition

Let $*$ be a continuous t-norm and $\mathrm{EP}(*)$ its endpoints. A $BL_{EP(*)}$-algebra **A** is a structure in the language of BL-algebras expanded with constants $\{a_r; r \in \mathrm{EP}(*)\}$ that validates all the axioms of $BL_{EP(*)}$ under $e(r) = a_r$ for all evaluations e.

$BL_{EP(*)}$-algebras are defined by a set of propositional formulas and therefore form a variety in the given language.
A standard $BL_{EP(*)}$-algebra is any $\mathrm{BL}_{\mathrm{EP}(*)}$-algebra on $[0, 1]$.

Lemma

Let $*$ be a continuous t-norm, $\mathrm{EP}(*)$ its endpoints, **A** *a $\mathrm{BL}_{\mathrm{EP}(*)}$-chain and $a_r = e(c_r)$ in* **A**. *Then for $r < s \in \mathrm{EP}(*)$ we have $a_r \leq a_s$ in* **A**; *if $a_r, a_s < 1$ in* **A**, *then $a_r < a_s$ in* **A**.

Translation functions

Given $*$, each $r \in \mathrm{EP}(*)$ defines a translation function $\varphi \longrightarrow \varphi^{[c_r, c_r{}^+]}$ on formulas of the language of BL:

$$\begin{aligned}
\overline{0}^{[c_r,c_r^+]} &= c_r\\
\overline{1}^{[c_r,c_r^+]} &= c_r^+\\
p^{[c_r,c_r^+]} &= (p \vee c_r) \wedge c_r^+\\
(\varphi \& \psi)^{[c_r,c_r^+]} &= \varphi^{[c_r,c_r^+]} \& \psi^{[c_r,c_r^+]}\\
(\varphi \rightarrow \psi)^{[c_r,c_r^+]} &= (\varphi^{[c_r,c_r^+]} \rightarrow \psi^{[c_r,c_r^+]}) \wedge c_r^+
\end{aligned}$$

Why translation functions?

Theorem

Let **A** *be any* $\mathrm{BL}_{\mathrm{EP}(*)}$*-chain s. t.* $r \in \mathrm{EP}(*)$*. Let* $\varphi \equiv \psi$ *be an arbitrary BL-equivalence. Then* $\varphi^{[c_r, c_r^+]} \equiv \psi^{[c_r, c_r^+]}$ *is a 1-tautology of* **A** *iff the BL-chain* $0 \oplus [a_r, a_r^+] \oplus 1$ *belongs to the variety generated by* $\mathrm{BL} \cup \{\varphi \equiv \psi\}$*.*

In particular, if φ is a tautology of $[0,1]_{Ł}$ but not of $[0,1]_{\mathrm{G}}$ or $[0,1]_{\Pi}$, then $[r, r^+]$ is an MV-component in **A** iff $\varphi^{[c_r, c_r^+]} \equiv c_r^+$ is a tautology of **A**. Similarly for G and Π.

Translating the axioms for components

(Ł) is the axiom $\neg\neg\varphi \to \varphi$ of Łukasiewicz logic,
(G) is the axiom $\varphi \to \varphi\&\varphi$ of Gödel logic, and
(Π) is the axiom $(\varphi \to \chi) \vee ((\varphi \to (\varphi \,\&\, \psi)) \to \psi)$ of product logic.

Given $*$ with rational endpoints $\mathrm{EP}(*)$, denote

$$\text{Ł}^{r} \text{ the formula } \text{Ł}^{[c_r, c_r^+]} \equiv c_r^+$$

$$\mathrm{G}^{r} \text{ the formula } \mathrm{G}^{[c_r, c_r^+]} \equiv c_r^+$$

$$\Pi^{r} \text{ the formula } \Pi^{[c_r, c_r^+]} \equiv c_r^+$$

The Logic $\mathrm{BL}_{\mathrm{COMP}(*)}$

Definition

Let $*$ be a continuous t-norm with rational endpoints EP. The logic $\mathrm{BL}_{\mathrm{COMP}(*)}$ has the axioms of $\mathrm{BL}_{\mathrm{EP}(*)}$, and in addition, for all $r \in \mathrm{EP}$ s. t. $r < r^{+}$:

$$\begin{array}{ll} (COMP^r_{Ł}) & Ł^r \text{ whenever } [r, r^{+}] \text{ in } [0,1]_* \text{ is a copy of } [0,1]_{Ł} \\ (COMP^r_{G}) & G^r \text{ whenever } [r, r^{+}] \text{ in } [0,1]_* \text{ is a copy of } [0,1]_{G} \\ (COMP^r_{\Pi}) & \Pi^r \text{ whenever } [r, r^{+}] \text{ in } [0,1]_* \text{ is a copy of } [0,1]_{\Pi} \end{array}$$

The deduction rule is modus ponens.

A (standard) $\mathrm{BL}_{\mathrm{COMP}(*)}$-algebra is a (standard) $\mathrm{BL}_{\mathrm{EP}(*)}$-algebra that satisfies all the axioms of $\mathrm{BL}_{\mathrm{COMP}(*)}$.

Completeness results

For any continuous t-norm $*$, the logic $\mathrm{BL}_{\mathrm{COMP}(*)}$ is strongly complete w. r. t. $\mathrm{BL}_{\mathrm{COMP}(*)}$-chains.

Theorem

Let $*$ *be a continuous t-norm. Let L be a* $\mathrm{BL}_{\mathrm{COMP}(*)}$*-chain. Then L is partially embeddable into a standard* $\mathrm{BL}_{\mathrm{COMP}(*)}$*-chain.*

It follows that the logic $\mathrm{BL}_{\mathrm{COMP}(*)}$ enjoys the finite strong standard completeness.

Theorem

If $[0,1]_*$ *is a finite ordinal sum of Ł-,* G*-, and Π-components, then the logic* $\mathrm{BL}_{\mathrm{COMP}(*)}$ *is complete w. r. t. the* $\mathrm{BL}_{\mathrm{COMP}(*)}$*-algebra* $[0,1]_*$ *(the canonical algebra).*

It is an open problem whether the same can be achieved, perhaps with some additional axioms, also for infinite sums

Computational complexity

Theorem

Let $*$ *be a continuous t-norm which is a finite ordinal sum. Let* $\mathbf{A}$ *be the* $\mathrm{BL}_{\mathrm{COMP}(*)}$*-algebra given by* $*$*. Then* $\mathrm{TAUT}(\mathbf{A})$ *is a co-NP-complete set.*

Obviously $\mathrm{TAUT}(\mathbf{A})$ is coNP-hard. To show containment, apply the algorithm for t-norm logics without constants.

Computational complexity

Theorem

Let $S \subseteq \mathbb{N}$ *and let* c_S *be its characteristic function. Let* $*$ *be a continuous t-norm with endpoints* $\{0, 1/2, 3/4, 7/8, \dots\}$*. Assume* $*$ *has two types of components Ł and* Π*, and for each* $i = 0, 1, \dots,$ *we have* $[(2^i - 1)/2^i, (2^{i+1} - 1)/2^{i+1}]$ *is an Ł-component iff* $i \in S$*. Let* $\mathbf{A}$ *be the* $\mathrm{BL}_{\mathrm{COMP}(*)}$*-algebra given by* $*$*. Then* S *is m-reducible to* $\mathrm{TAUT}(\mathbf{A})$*.*