Dempster-Shafer Degrees of Belief in Łukasiewicz Logic

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The Framework

Combining degrees of truth and degrees of belief

Φ...set of formulas in Łukasiewicz logic
$p : \Phi \to [0, 1]$

Study of p as a

- **probability** (Mundici, Riečan, Di Nola...)
- **necessity/possibility** (Flaminio, Godo, Marchioni)
- **coherent lower probability/coherent upper probability** (Fedel, Keimel, Montagna, Roth)
- **belief/plausibility**

Game-theoretic Interpretation

Bookmaking over infinite-valued events

De Finetti's coherence criterion for bets on MV events like

"Manchester United will score early"

(Mundici, Montagna, ...)

Cooperative games

- coalition = principle φ
- player's degree of membership to a coalition = level of conformity of the player with φ

(Butnariu, TK)

Averaging the Truth Value in Ł

- φ formula in Łukasiewicz logic
- V_1, V_2 truth valuations
- s_c "averaged" truth valuation:

$$s_c(\varphi) := cV_1(\varphi) + (1-c)V_2(\varphi), \quad c \in [0,1]$$

- (s_n) convergent (in $[0,1]^{L_k}$) sequence of "averaged" TVs:

$$s(\varphi) := \lim_{n\to\infty} s_n(\varphi)$$

Definition (Mundici, Riečan)

A state s on L_k is a function $L_k \to [0,1]$ with $s(0) = 0, s(1) = 1$ and

$$s(f \oplus g) = s(f) + s(g), \text{ for every } f, g \in L_k \text{ s.t. } f \odot g = 0.$$

Averaging the Truth Value in Ł (ctnd.)

Theorem

Every state on L_k is an integral w.r.t. some regular Borel probability measure on possible worlds $[0,1]^k$.

Theorem

Let s be a state on L_k. Then the FAE:

- *s is an extreme point of $S(L_k)$*
- *there exists $x \in [0,1]^k$ with $s = s_x$*
- *s is a homomorphism into the standard MV-algebra $[0,1]$*
- *the set $\{f \in L_k \mid s(f) = 1\}$ is a maximal filter*
- *the set $\{f \in L_k \mid s(f) = 0\}$ is a maximal ideal*

Averaging the Relative Truth Value in Ł

- φ formula in Łukasiewicz logic
- A nonempty closed (in $[0,1]^k$) set of truth valuations
- Pavelka-style truth degree of φ over A:

$$\|\varphi\|_A := \inf\{\, V(\varphi) \mid V \in A \,\}$$

Question

Which function on L_k is obtained by

- averaging $\|\varphi\|_{A_1}, \|\varphi\|_{A_2}$
- taking limits of such averages

?

Generalizing States

MV probability	**MV belief functions**
state on L_k	belief function on L_k
Lebesgue integral	?
representing probability measure	?
$s_x(f) = f(x)$	?
$x \in [0,1]^k$	?
truth valuation	?
maximal filter	?
truth degree of φ	?

Belief Function

- function with a nonnegative Möbius transform (G.-C. Rota)
- totally monotone function

Generalizations

Belief functions can be studied on lattices

BUT

we are especially interested in connections to Łuk. logic/algebra

L. Godo, P. Hájek, and F. Esteva.
A fuzzy modal logic for belief functions.
Fund. Inform., 57(2-4):127–146, 2003.

Belief Function (ctnd.)

Definition (Dempster, Shafer)

Let X be a finite nonempty set. A function

$$\beta : \mathcal{P}(X) \to [0, 1]$$

is a belief measure if there is a mapping (basic assignment)

$$m : \mathcal{P}(X) \to [0, 1]$$

with $m(\emptyset) = 0$ and $\sum_{A \in \mathcal{P}(X)} m(A) = 1$ such that

$$\beta(A) = \sum_{B \subseteq A} m(B), \quad A \in \mathcal{P}(X).$$

Examples

Example (Heads or tails or out of sight?)

$X = \{H, T\}$

$$m(A) = \begin{cases} h, & A = \{H\} \\ t, & A = \{T\} \\ 1 - h - t, & A = X \end{cases} \quad h + t < 1, h, t \geqslant 0$$

Example (Principle of insufficient reason)

$$m(A) = \begin{cases} 1, & A = X \\ 0, & \text{otherwise} \end{cases}$$

Total Monotonicity

Theorem

The FAE:

1. β *is a belief measure*
2. $\beta : \mathcal{P}(X) \to [0,1]$ *satisfies* $\beta(\emptyset) = 0$, $\beta(X) = 1$ *and*
   - *it is monotone*
   - *for each* $n \geqslant 2$ *and every* $A_1, \ldots, A_n \in \mathcal{P}(X)$*:*

$$\beta\left(\bigcup_{i=1}^{n} A_i\right) \geqslant \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \beta\left(\bigcap_{i \in I} A_i\right).$$

The function m_β constructed in (2) $\Rightarrow$ (1) is called the Möbius transform of β.

Belief Measures: From BAs to MVs

Belief measures	**Belief functions**
belief measure on $\mathcal{P}(X)$	belief function on L_k
basic assignment	?
TM set function	?

- the mapping $A \mapsto \{B \in \mathcal{P}(X) \mid B \subseteq A\}$ sends the event A to a set of all sets of possible worlds rendering A true:

$$\|A\|_B := \min\{\, A(x) \mid x \in B \,\}$$
$$\|A\| = \{\, B \in \mathcal{P}(X) \mid B \subseteq A \,\}$$

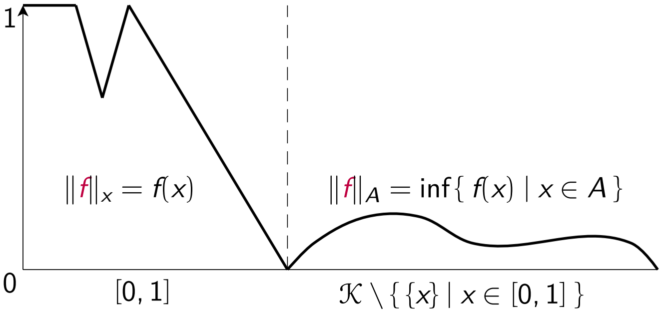
- **belief** of A = **probability** of $\|A\|$:

$$\beta(A) = m(\|A\|)$$

Belief Measures: From BAs to MVs (ctnd.)

- $f \in L_k$, k-variable McNaughton function
- $A \in \mathcal{K}$, nonempty closed subset of $[0,1]^k$
- define $\|f\|_A := \inf\{\, f(x) \mid x \in A \,\}$
- **belief** of f = **state** of $\|f\|$:

$$\mathrm{Bel}(f) = \mathbf{s}(\|f\|)$$



Duality

Cignoli, D'Ottaviano, Mundici (2000):

Theorem (From sets to filters and back)

$A \mapsto F_A := \{ f \in L_k \mid f(x) = 1, x \in A \}$ determines an order-reversing bijection between the nonempty ***closed subsets*** *A of $[0,1]^k$ and the* ***proper filters*** *in L_k that are intersections of maximal filters.*

Theorem (From theories to filters and back)

$\Theta \mapsto \{ f_\varphi \mid \varphi \in \Theta \}$ determines a bijection between nonempty ***deductive theories*** *Θ in Ł such that $\Theta = \Theta^{\models}$ and the* ***proper filters*** *in L_k that are intersections of maximal filters.*

$\|f\|_A$ can be viewed as the truth degree of φ_f over Θ_A:

$$\|f\|_A = \inf \{ V(\varphi_f) \mid V \text{ is a model of } \Theta_A \}$$

Space of Closed Subsets

Definition

Let $\mathcal{K}$ be the set of all nonempty closed subsets of $[0,1]^k$ equipped with the Hausdorff metric d_H given by

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}, \quad A,B\in\mathcal{K}.$$

Theorem

The metric space $(\mathcal{K}, d_H)$ is ***compact****.*

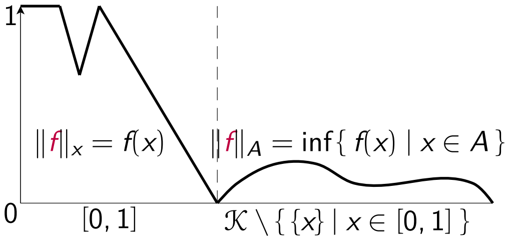
Continuation of McNaughton Functions

$C(\mathcal{K})$ the MV-algebra of all continuous functions $\mathcal{K} \to [0,1]$

Proposition

The mapping $\|\cdot\| : L_k \to [0,1]^{\mathcal{K}}$ is

- into $C(\mathcal{K})$
- injective
- preserving any existing infima from L_k to $C(\mathcal{K})$



Belief Functions

A state assignment is any state on $C(\mathcal{K})$.

Definition

Let $\mathbf{s}$ be a state assignment on $C(\mathcal{K})$. A belief function is a mapping $\mathrm{Bel} : L_k \to [0,1]$ given by

$$\mathrm{Bel}(f) = \mathbf{s}(\|f\|), \quad f \in L_k.$$

Example

For each $A \in \mathcal{K}$, the function $\mathrm{Bel}_A(f) = \|f\|_A$ is a belief function whose state assignment is $\mathbf{s}_A$, where

$$\mathbf{s}_A(h) = h(A), \quad h \in C(\mathcal{K}).$$

Properties

Proposition

Let Bel be a belief function on L_k. Then:

- $\mathrm{Bel}(0) = 0$, $\mathrm{Bel}(1) = 1$
- if $f \odot g = 0$, then $\mathrm{Bel}(f \oplus g) \geqslant \mathrm{Bel}(f) + \mathrm{Bel}(g)$
- $\mathrm{Bel}(f) + \mathrm{Bel}(\neg f) \leqslant 1$
- Bel is a state if the state assignment **s** "lives" only on L_k
- Bel is **totally monotone** on the lattice reduct of L_k:
  - it is monotone
  - for each $n \geqslant 2$ and every $f_1, \ldots, f_n \in L_k$:

$$\mathrm{Bel}\left(\bigvee_{i=1}^{n} f_i\right) \geqslant \sum_{\substack{I \subseteq \{1,\ldots n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \, \mathrm{Bel}\left(\bigwedge_{i \in I} f_i\right).$$

Representation of BFs

Theorem

For every belief function Bel *on* L_k *there exists a unique*

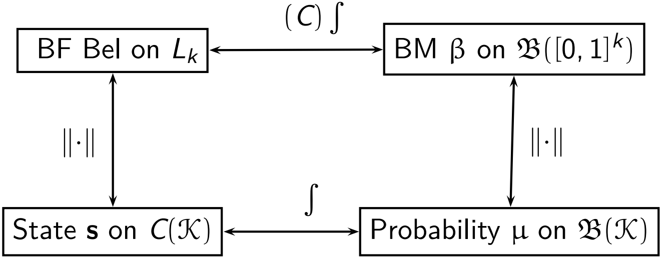
1. ***Borel probability measure*** μ *on* $\mathfrak{B}(\mathcal{K})$ *such that*

$$\mathrm{Bel}(f) = \int_{\mathcal{K}} \|f\| \,\mathrm{d}\mu, \quad f \in L_k$$

2. ***belief measure*** β *on* $\mathfrak{B}([0,1]^k)$ *such that*

$$\mathrm{Bel}(f) = (C) \int_{[0,1]^k} f \,\mathrm{d}\beta, \quad f \in L_k$$

Scheme


BF Bel on L_k
$(C)\int$
BM β on $\mathfrak{B}([0,1]^k)$
$\|\cdot\|$
$\|\cdot\|$
$\int$
State $\mathbf{s}$ on $C(\mathcal{K})$
Probability μ on $\mathfrak{B}(\mathcal{K})$

Space of Belief Functions

Theorem

Let Bel *be a belief function on* L_k*. Then the FAE:*

- Bel *is an extreme point of* $BEL(L_k)$
- *there exists* $A \in \mathcal{K}$ *with* $\mathrm{Bel} = \mathrm{Bel}_A$
- $\{f \in L_k \mid \mathrm{Bel}(f) = 1\}$ *is a filter that is* $\bigcap$ *of maximal filters*
- $\{f \in L_k \mid \mathrm{Bel}(f) = 0\}$ *is an ideal that is* $\bigcap$ *of maximal ideals*

Compare:

D. Mundici.
Averaging the truth-value in Łukasiewicz logic.
Studia Logica, 55(1):113–127, 1995.

BF as a Lower Probability

In the spirit of:

M. Fedel, K. Keimel, F. Montagna, and W. Roth.
Imprecise probabilities, bets and functional analytic methods in Łukasiewicz logic.
Submitted, 2010.

Theorem

For every $f \in L_k$:

$$\mathrm{Bel}(f) = \min\left\{ s(f) \mid s \text{ state on } L_k \text{ with } s \geqslant \mathrm{Bel} \right\}$$

Necessity/Possibility Functions

In Dempster-Shafer theory necessity measures are those BFs for which the mass assignment is supported by a chain.

Definition

A necessity function on L_k is a BF Bel such that the corresponding state assignment on $C(\mathcal{K})$ "lives" on a chain in $\mathcal{K}$.

Warning: $\mathrm{Bel}(f \wedge g) \neq \mathrm{Bel}(f) \wedge \mathrm{Bel}(g)$

Generalization

All the results for BFs on the free k-generated free MV-algebra remain valid for any MV-algebra with the second-countable maximal ideal space:

TK
Generalized Möbius Transform of Games on MV-algebras and Cimmino-type Algorithm for Core
Submitted to CONM/IMCP series of the AMS

Algebraic and Other Issues

- Which smaller MV-algebra can be used in place of $C(\mathcal{K})$?

  Precisely: describe the MV-algebra generated by

  $$\{ \|f\| \mid f \in L_k \}$$

- **Behavioral interpretation** of BFs (with T. Flaminio)