

# Kripke-Style Semantics for Normal Systems

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# Normal Gentzen Systems

- Fully-structural propositional sequential systems
- Normal derivation rule:
  - The rule:  $s_1 s_2 \dots s_m / c$
  - Its application:

$$\frac{\sigma(s_1)+\text{context}_1 \dots \sigma(s_m)+\text{context}_m}{\sigma(c)+\text{context}}$$

# Normal Rules

- Example

$$P_1 \Rightarrow P_2 / \Rightarrow P_1 \supset P_2$$

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$$\frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

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# Normal Rules

- Example

$$\frac{P_1 \Rightarrow P_2 / \Rightarrow P_1 \supset P_2}{\text{wff} \Rightarrow \text{wff}}$$

$$\frac{P_1 \Rightarrow P_2 / \Rightarrow P_1 \supset P_2}{\text{wff} \Rightarrow \emptyset}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \supset \psi}$$

- Each premise is associated with a context restriction.

# Many-Sided Sequents

- Classical sequent:  $\Gamma \Rightarrow \Delta$  ( $\Gamma, \Delta$  are finite **sets** of formulas)
- n-sided sequent:  $\Gamma_1 ; \Gamma_2 ; \dots ; \Gamma_n$

- Notation

- Let  $\mathcal{I}$  be a finite set of signs.
- A signed formula:  $i \div \psi$  ( $i \in \mathcal{I}$  and  $\psi$  is a formula).
- A sequent is a finite set of signed formulas.

- Negative interpretation [Baaz, Fermüller, Zach '93]

$\{i_1 \div \psi_1, \dots, i_n \div \psi_n\}$  is true iff  
 $i_k$  is **not** the truth-value assigned to  $\psi_k$  for some  $1 \leq k \leq n$

- Classical sequent notation  $\mathcal{I} = \{t, f\}$

$\Gamma \Rightarrow \Delta$  is  $\{t \div \psi \mid \psi \in \Gamma\} \cup \{f \div \psi \mid \psi \in \Delta\}$

# Normal Rules

- A normal rule:
  - **Premises:** sequents  $s_1 \dots s_m$
  - **Context restrictions:** corresponding sets of signed formulas  $\pi_1 \dots \pi_m$
  - **A Conclusion:** a sequent  $c$
- Its application:

$$\frac{\sigma(s_1) \cup (s' \cap \pi_1) \quad \dots \quad \sigma(s_m) \cup (s' \cap \pi_m)}{\sigma(c) \cup s'}$$

# Examples

- S4's Box

$$\mathcal{I} = \{t, f\}$$

$$\langle \Rightarrow p_1, \pi \rangle / \Rightarrow \Box p_1$$

$$\pi = \{t \div \Box \varphi \mid \varphi \in \text{wff}\} \cup \{f \div \Diamond \varphi \mid \varphi \in \text{wff}\}$$

$$\frac{\Box \Gamma \Rightarrow \Diamond \Delta, \psi}{\Box \Gamma \Rightarrow \Diamond \Delta, \Box \psi}$$

- 3-Valued Lukasiewicz's Implication [Zach '93]

$$\mathcal{I} = \{f, l, t\}$$

$$\langle f \div p_1, \pi \rangle, \langle l \div p_1, l \div p_2, \pi \rangle, \langle t \div p_2, \pi \rangle / t \div p_1 \supset p_2$$

$$\pi = \{i \div \varphi \mid i \in \mathcal{I}, \varphi \in \text{wff}\} \quad (\text{i.e. no context restriction})$$

$$\frac{\Gamma, \varphi; \Delta; \Sigma \quad \Gamma; \Delta, \varphi, \psi; \Sigma \quad \Gamma; \Delta; \Sigma, \psi}{\Gamma; \Delta; \Sigma, \varphi \supset \psi}$$

# Normal Gentzen Systems

- Axioms

$$\frac{}{i \div \varphi, j \div \varphi}$$

for every  $i \neq j$  in  $\mathcal{I}$

- Weakenings

$$\frac{s}{s, i \div \varphi}$$

for every  $i \in \mathcal{I}$

- Cut

$$\frac{s, i_1 \div \varphi \quad \dots \quad s, i_n \div \varphi}{s}$$

where  $\mathcal{I} = \{i_1, \dots, i_n\}$

- Finite set of normal Gentzen rules

# Examples of Normal Gentzen Systems

- Normal Gentzen systems are suitable for:
  - Classical logic
  - Intuitionistic logic
  - Dual-intuitionistic logic
  - Bi-intuitionistic logic
  - S4 and S5
  - n-valued Lukasiewicz logic
  - ...

# Kripke Semantics

- Kripke Frame
  - Set of worlds  $W$
  - *Legal* set of relations  $R_1, R_2, \dots \subseteq W \times W$
  - *Legal* valuation  $v : W \times \text{wff} \rightarrow \mathcal{I}$
- A sequent  $s$  is *true* in  $a \in W$  iff  $\exists i \div \varphi \in s . v(a, \varphi) \neq i$
- A sequent is *R-true* in  $a \in W$  iff it is true in every  $b \in W$  s.t.  $aRb$

# G-Legal Kripke Frames

- A normal system  $G$  induces a set of **G-legal** Kripke frames:
  - A preorder  $R_\pi$  for every context-restriction  $\pi$  in  $G$ .
    - If  $\pi$  is the set of all signed formulas,  $R_\pi$  is the identity relation.
  - Generalized persistence condition  
If  $i \div \varphi \in \pi$ ,  $v(a, \varphi) = i$  implies  $v(b, \varphi) = i$  for every  $b$  s.t.  $a R_\pi b$ .
  - The valuation should *respect* the logical rules of  $G$   
Respect a rule  $\langle s_1, \pi_1 \rangle, \dots, \langle s_m, \pi_m \rangle / c$  :  
If  $\forall 1 \leq i \leq m. \sigma(s_i)$  is  **$R_{\pi_i}$ -true** in  $a \in \mathbb{W}$ , then  $\sigma(c)$  is **true** in  $a$

# Example: Primal Intuitionistic Implication (Gurevich '09)

- Rules

1	$\langle \Rightarrow p_2, \text{wff} \Rightarrow \emptyset \rangle / \Rightarrow p_1 \rightsquigarrow p_2$	$\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \rightsquigarrow \psi}$
2	$\langle \Rightarrow p_1, \text{wff} \Rightarrow \text{wff} \rangle, \langle p_2 \Rightarrow, \text{wff} \Rightarrow \text{wff} \rangle / p_1 \rightsquigarrow p_2 \Rightarrow$	$\frac{\Gamma \Rightarrow \Delta, \phi \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \rightsquigarrow \psi \Rightarrow \Delta}$

- Semantics

$$\leq = R_{\text{wff} \Rightarrow \emptyset}$$

Persistence	if $v(a, \phi) = t$ then $v(b, \phi) = t$ for every $b \geq a$
Respect rule #1	$v(a, \phi \rightsquigarrow \psi) = t$ if $v(b, \psi) = t$ for every $b \geq a$
Respect rule #2	$v(a, \phi \rightsquigarrow \psi) = f$ if $v(a, \phi) = t$ and $v(a, \psi) = f$

- $v(a, \phi \rightsquigarrow \psi)$  is **free** when  
 $v(a, \psi) = f$  and  $(v(b, \phi) = f \text{ or } v(b, \psi) = t \text{ for every } b \geq a)$

# Strong Soundness and Completeness

- For every normal system  $G$ ,  
 $C \vdash_G s$  iff every  $G$ -legal frame which is a model of  $C$  is also a model of  $s$ .
- Applications:
  - Several known completeness theorems are immediately obtained as special cases.
  - Compactness theorem.
  - Basis for semantic proofs of analyticity.
  - Semantic decision procedure for analytic systems.

# Analycity

- A normal system is **(strongly) analytic** iff it has the subformula property, i.e.  
 $C \vdash_G s$  implies that there exists a proof of  $s$  from  $C$  in  $G$  that contains only subformulas of  $C$  and  $s$ .
- Analycity implies:  
decidability, consistency and conservativity.

# Semantic Characterization of Analycity

- **Def** Given a set  $E$  of formulas, an **E-Semiframe** is a Kripke frame, except for:  $v : W \times E \rightarrow I$  instead of  $v : W \times \text{wff} \rightarrow I$ .
- **Notation** Given a set  $E$  of formulas,  $C \vdash_G^E s$  iff there exists a proof of  $s$  from  $C$  in  $G$  containing only formulas from  $E$ .
- **Thm**  $C \vdash_G^E s$  iff every  $G$ -legal **E-semiframe** which is a model of  $C$  is also a model of  $s$ .
- **Cor**  $G$  is analytic **iff** for every  $C$  and  $s$ , if every  $G$ -legal frame which is a model of  $C$  is also a model of  $s$ , then this is also true with respect to  $\text{sub}[C,s]$ -semiframes.

# Semantic Proofs of Analycity

- To prove that  $G$  is analytic, it suffices to prove:  
Every  $G$ -legal semiframe can be extended to a  $G$ -legal frame.
- This can be easily done in many important cases:
  - Normal systems for classical logic, intuitionistic logic, dual and bi-intuitionistic logics, S4, S5, n-valued Lukasiewicz's logic...
  - The family of coherent canonical systems.

# Normal Hypersequent Systems

- This approach can be extended to hypersequent systems (with internal contraction).
- Generalized communication rule:

$$\frac{G \mid s_1 \cup (s_2 \cap \pi) \quad G \mid s_2 \cup (s_1 \cap \pi)}{G \mid s_1 \mid s_2}$$



$R_\pi$  is linear

- Examples:                      S4.3                      Gödel logic

# Application: Adding Non-Deterministic Connectives to Gödel Logic

- General Motivations:
  - Uncertainty or ambiguity on the level of the connectives
  - Gurevich's Motivation: proof-search complexity

- Examples

- $v(\varphi \rightsquigarrow \psi) \in [v(\psi), v(\varphi) \rightarrow v(\psi)]$

$$\frac{G \mid \Gamma \Rightarrow \Delta, \varphi \quad G \mid \Gamma, \psi \Rightarrow \Delta}{G \mid \Gamma, \varphi \rightsquigarrow \psi \Rightarrow \Delta}$$

$$\frac{G \mid \Gamma \Rightarrow \psi}{G \mid \Gamma \Rightarrow \varphi \rightsquigarrow \psi}$$

- $v(\varphi \bowtie \psi) \in [\min\{v(\varphi), v(\psi)\}, \max\{v(\varphi), v(\psi)\}]$

$$\frac{G \mid \Gamma, \varphi \Rightarrow \Delta \quad G \mid \Gamma, \psi \Rightarrow \Delta}{G \mid \Gamma, \varphi \bowtie \psi \Rightarrow \Delta}$$

$$\frac{G \mid \Gamma \Rightarrow \Delta, \varphi \quad G \mid \Gamma \Rightarrow \Delta, \psi}{G \mid \Gamma \Rightarrow \Delta, \varphi \bowtie \psi}$$

- Remains decidable

**Thank you!**