

Linearity issues in the algebra of Łukasiewicz logic

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MV-algebras

- ▶ MV-algebras are the algebraic structures of the ∞ -valued Łukasiewicz logic.
- ▶ (C.C.Chang, 1958) *MV-algebra* $(A, \oplus, *, 0)$
 1. $(A, \oplus, 0)$ abelian monoid,
 2. $(x^*)^* = x$,
 3. $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
 4. $0^* \oplus x = 0^*$.
- ▶ $1 := 0^*$, $x \odot y := (x^* \oplus y^*)^*$, $x \vee y := (y^* \oplus x)^* \oplus x$,
 $x \wedge y := (x^* \vee y)^*$
- ▶ Standard model: $([0, 1], \oplus, *, 0)$
 $x \oplus y = \min(x + y, 1)$, $x \odot y = \max(x + y - 1, 0)$,
 $x^* = 1 - x$, $1 = 0^*$, $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$
- ▶ A Boolean algebra iff $\oplus = \vee$ and $\odot = \wedge$.

MV-algebras and lattice-ordered groups

C.C.Chang, 1958

An **MV-algebra** is a structure $(A, \oplus, *, 0)$ such that:

1. $(A, \oplus, 0)$ abelian monoid,
2. $(x^*)^* = x$,
3. $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
4. $0^* \oplus x = 0^*$.

Mundici, 1986

For any MV-algebra A there exists an abelian lattice-ordered group with strong unit (G, u) such that $A \simeq [0, u]_G$.

- ▶ $(G, +, 0, \leq)$ lattice-ordered group: $(G, +, 0)$ group, (G, \leq) lattice, $x \leq y$ implies $x + z \leq y + z$ for any $x, y, z \in G$.
- ▶ u strong unit: $u \geq 0$, for any $x \in G$ there is $n \geq 1$ s.t. $x \leq nu$.
- ▶ $\Gamma(G, u) = ([0, u]_G, \oplus, *, 0)$: $x \oplus y = (x + y) \wedge 1$, $x^* = 1 - x$.

What linearity means in the theory of MV-algebras?

- ▶ $x, y \in A = [0, u]_G$
 $x \oplus_A y = x +_G y$ iff $x \leq y^*$ iff $x \odot y = 0$
- ▶ If A, B MV-algebras then $\omega : A \rightarrow B$ is linear if $\omega(x \oplus y) = \omega(x) \oplus \omega(y)$ whenever $x \odot y = 0$.
- ▶ Homogeneity.
If rx and ry exists for any $x \in A, y \in B$ and $r \in [0, 1]$ then ω also satisfies $\omega(rx) = r\omega(y)$. We have a similar property for $([0, 1] \cap \mathbb{Q})$.

States on MV-algebras (Mundici, 1995)

- ▶ Let A be an MV-algebra. A state is a linear function $s : A \times [0, 1]$ such that $s(1_A) = 1$.
- ▶ if $A = [0, u]_G$ then there exists a unique state $\bar{s} : G \rightarrow \mathbb{R}$ s.t. $\bar{s}(x) = s(x)$ whenever $x \in A$
- ▶ if A is a Boolean algebra then s is a Boolean probability.

- ▶ The theory of MV-algebras supports an intrinsic notion of linearity, which is rooted in the Boolean probability theory and in the theory of states defined on lattice ordered groups. For an insight on the probabilistic view on states one can see [D. Mundici, B. Riečan, Probability in MV-algebras, Handbook of Measure Theory, 2002.](#)

Lukasiewicz rings (Belluce and Di Nola, 2009)

- ▶ The aim was to characterize those commutative rings whose lattice of ideals enriched with the ideal product is an MV-algebra. Such a ring was called *Lukasiewicz ring*.
- ▶ If R is a commutative ring, define th structure $A(R) = (Id(R), \oplus, \odot, *, \mathbf{0}, \mathbf{1})$ as follows:
 $\mathbf{0} = \{0\}$, $\mathbf{1} = R$, $I \oplus J := (I^* J^*)^*$, $I \odot J := IJ$,
 $I^* := ann(I) = \{x \in R \mid xy = 0 \ \forall y \in I\}$.

Theorem.

Tfae:

- ▶ $A(R)$ is an MV-algebra,
- ▶ R is an *Lukasiewicz ring*, i.e.
 - for any $x \in R$ there exists $y \in R$ s.t. $xy = x$,
 - $I + J = (I^*(I^* J)^*)^*$ for any $I, J \in Id(R)$,
- ▶ R is a direct sum of Artinian chain rings with unit.

MV-algebras and Bézout domains (Rump, Yang, 2010)

- ▶ A Bézout domain is an integral domain such that any finitely generated ideal is principal. The group of divisibility associated to a Bézout domain is lattice ordered. The authors goal was to characterize those Bézout domains with the property that the group of divisibility has a strong unit.
- ▶ Let D be a integral domain. If K is the quotient field of D , U the set of units of D , $K^* = K \setminus \{0\}$, $D^* = D \setminus \{0\}$ then $\mathcal{G}(D) := K^*/U$ is the group of divisibility. If we set $aU \leq bU$ iff $b/a \in D$ then $(\mathcal{G}(D), \leq)$ is a directed p.o. group, $\mathcal{G}(D)^+ = D^*/U$.
- ▶ The group of divisibility has proved useful to phrase ring-theoretic problems of D in terms of of the ordered group $\mathcal{G}(D)$.

- ▶ (Jaffard-Ohm) An abelian p.o. group G is lattice ordered iff there exists a Bézout domain D such that $G \simeq \mathcal{G}(D)$.
- ▶ (Yang and Rump, 2010) Tfae:
 1. $\mathcal{G}(D)$ has a strong unit,
 2. $\bigcap \{P \subseteq D \mid \{0\} \neq P \text{ prime ideal}\} \neq \{0\}$.

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- ▶ (Yang and Rump, 2010) Tfae:
 1. $\mathcal{G}(D)$ has a strong unit,
 2. $\bigcap \{P \subseteq D \mid \{0\} \neq P \text{ prime ideal}\} \neq \{0\}$.
- ▶ Any MV-algebra is isomorphic with the unit interval of $\mathcal{G}(D)$ for some Bézout domain D satisfying $\bigcap \{P \subseteq D \mid \{0\} \neq P \text{ prime ideal}\} \neq \{0\}$.
- ▶ It is well-known that statements about MV-algebras can be expressed as statements about lattice-ordered groups. The above result one can phrase ring-theoretical problems in the language of MV-algebras and vice versa.

- ▶ MV-algebras and rings

- ▶ MV-algebras and rings
- ▶ MV-algebras and Riesz spaces

Extensions of Mundici's equivalence

- ▶ Mundici, 1986
MV-algebras \simeq Lattice-ordered groups with strong unit
 $A \simeq [0, u]_G$
- ▶ Di Nola, Dvurecenskij, 2001
PMV-algebras \simeq Lattice-ordered rings with strong unit
 $P \simeq [0, u]_Q$
- ▶ Di Nola, Flondor, L., 2003
Riesz MV-algebras \simeq Lattice-ordered real vector spaces with strong unit
 $R \simeq [0, u]_V$

Riesz MV-algebras are MV-modules over $[0, 1]$. The categorical equivalence was proved in the general context of MV-modules over a fixed PMV-algebra.

Riesz spaces (vector lattices)

A Riesz space Q is a lattice-ordered real vector space such that:

1. $x \leq y$ implies $x + z \leq y + z$,
 2. $r \geq 0$ and $x \geq 0$ implies $rx \geq 0$,
- for any $x, y, z \in Q$ and $r \in [0, 1]$.

Riesz MV-algebras

A Riesz MV-algebra is a pair (R, \cdot) where R is an MV-algebra and $\cdot : [0, 1] \times R \times R$ satisfying the following properties:

1. $r \cdot (x \oplus y) = (r \cdot x) \oplus (r \cdot y)$ whenever $x \odot y = 0$,
2. $(r \oplus q) \cdot x = (r \cdot x) \oplus (q \cdot x)$ whenever $r \odot q = 0$,
3. $r \cdot (q \cdot x) = (rq) \cdot x$,
4. $1 \cdot x = x$,

for any $x, y \in R$ and $q, r \in [0, 1]$.

- ▶ For any MV-algebra R there exists a Riesz space with strong unit (Q, u) such that $R \simeq [0, u]_Q$.
- ▶ The class of Riesz MV-algebras is equational and it is generated by $[0, 1]$.

Assume R semisimple Riesz MV-algebra

$$\bigcap \{M \subseteq R \mid M \text{ maximal ideal}\} = \{0\}.$$

Definition

A *norm* on R is a function $\rho : M \rightarrow [0, 1]$ such that:

1. $\rho(1) = 1$, $\rho(x) = 0$ iff $x = 0$,
2. $\rho(x \oplus y) \leq \rho(x) \oplus \rho(y)$,
3. $\rho(rx) = r\rho(x)$,
4. $x \leq y$ implies $\rho(x) \leq \rho(y)$.

The norm ρ defines a metric $d_\rho(x, y) := \rho(d(x, y))$, where $d(x, y) = (x \odot y^*) \oplus (x^* \odot y)$.

We say that (R, ρ) is norm-complete if the metric space (R, d_ρ) is Cauchy complete.

Norm-complete Riesz MV-algebras

- ▶ $\|x\|_u = \inf \{r \in [0, 1] \mid x \leq r \cdot 1\}$ for any $x \in R$ (unit norm)
 $\|x \vee y\|_u = \|x\|_u \vee \|y\|_u$ for any $x, y \in R$
- ▶ $s: R \rightarrow [0, 1]$ state with $s^{-1}(0) = \{0\}$ (faithful state)
 $s(x \oplus y) = s(x) + s(y)$ whenever $x \odot y = 0$ (additive norm).
- ▶ We characterize R such that $(R, \|\cdot\|_u)$, respectively (R, s) for a given s , is norm-complete.

$(R, \|\cdot\|_u)$ (joint work with Enzo Marra)

Theorem

A Riesz MV-algebra R is norm-complete w.r.t. $\|\cdot\|_u$ iff $R \simeq C(\text{Max}(R), [0, 1])$, where $\text{Max}(R)$ is the compact Hausdorff space of the maximal ideals of R .

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(Ann.Math. 1941) Kakutani Theorem

For any M -space with strong unit (M, u) there exists a compact Hausdorff space X such that (M, u) is isometrically Riesz isomorphic with $C(X, \mathbb{R})$.

A M -space is a norm-complete Riesz space with a \vee -preserving norm.

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Theorem

The category of Riesz MV-algebra which are norm-complete w.r.t $\|\cdot\|_u$ with Riesz MV-algebra homomorphisms is dually equivalent with the category of M -spaces with strong unit with unital Riesz-space homomorphisms.

Theorem

The category of Riesz MV-algebra which are norm-complete w.r.t $\|\cdot\|_u$ and Riesz MV-algebra homomorphisms is dually equivalent with the category of compact Hausdorff spaces and continuous functions.

- ▶ **KHaus**: compact Hausdorff spaces and continuous functions
- ▶ **CRMV $_u$** : norm-complete Riesz MV-algebras and (Riesz) MV-algebra homomorphisms

- ▶ $\mathcal{C}: \mathbf{KHaus} \rightarrow \mathbf{CRMVu}$,
 $\mathcal{C}X = C(X, [0, 1])$,
 $\mathcal{C}\phi: \mathcal{C}X_2 \rightarrow \mathcal{C}X_1$,
 $\mathcal{C}\phi(f) = f \circ \phi$ for any $f \in \mathcal{C}X_2$ and
any continuous function $\phi: X_1 \rightarrow X_2$.

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any continuous function $\phi: X_1 \rightarrow X_2$.
- ▶ $Max: \mathbf{CRMVU} \rightarrow \mathbf{KHaus}$,
 $MaxR$ with hull-kernel topology,
 $Maxh: MaxR_2 \rightarrow MaxR_1$,
 $Maxh(m) = h^{-1}(m)$ for any $m \in MaxR_2$ and
any MV-homomorphism $h: R_1 \rightarrow R_2$.

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any MV-homomorphism $h: R_1 \rightarrow R_2$.
- ▶ \mathcal{C} and Max establish a categorical equivalence.

(R, s) where s is a state (IL)

(Ann.Math. 1941) Kakutani theorem

For any **L-space with strong unit** (L, u) there exists a measure space (X, Ω, μ) such that L is isometrically Riesz isomorphic with $L^1(\mu)$.

An *L-space* is a norm-complete Riesz space with an additive norm.

Definition

Let R be a Riesz MV-algebra and $s : R \rightarrow [0, 1]$ a faithful state. We say that (A, s) is *state-complete* if it is norm-complete w.r.t. the norm s . If (A, s) and (B, p) are MV-algebras with states, an MV-algebra homomorphism $h : A \rightarrow B$ is *state-preserving* if $p \circ h = s$.

Theorem

Let R be a Riesz MV-algebra and $s : R \rightarrow [0, 1]$ a faithful state. Then R is norm-complete w.r.t. s iff there exists a measure space (X, Ω, μ) such that $R \simeq L^1(\mu)_u$, where $L^1(\mu)_u := [\mathbf{0}, \mathbf{1}]_{L^1(\mu)}$.

Kroupa-Panti Theorem

Let A be an MV-algebra. For any state $s : A \rightarrow [0, 1]$, there exists a unique regular measure μ defined on the Borel σ -algebra of $X = \text{Max}(A)$ such that $s(f) = \int_X f \, d\mu$ for any $f \in A$.

The measure μ is *regular* if it is finite and for every Borel set B and every $\epsilon > 0$ there exists an open subset U of X such that $A \subset U$ and $\mu(U) \leq \mu(A) + \epsilon$.

Theorem

Let R be a Riesz MV-algebra and $s : R \rightarrow [0, 1]$ a faithful state. Then R is norm-complete w.r.t. s iff there exists a measure space (X, Ω, μ) such that $R \simeq L^1(\mu)_u$, where $L^1(\mu)_u := [\mathbf{0}, \mathbf{1}]_{L^1(\mu)}$.

- ▶ $X = \text{Max}(A)$, Ω is the σ -algebra of Borel sets of X and μ is the unique regular measure corresponding to s .

Kroupa-Panti Theorem

Let A be an MV-algebra. For any state $s : A \rightarrow [0, 1]$, there exists a unique regular measure μ defined on the Borel σ -algebra of $X = \text{Max}(A)$ such that $s(f) = \int_X f d\mu$ for any $f \in A$.

The measure μ is *regular* if it is finite and for every Borel set B and every $\epsilon > 0$ there exists an open subset U of X such that $A \subset U$ and $\mu(U) \leq \mu(A) + \epsilon$.

- ▶ Def. A space (X, Ω, μ) is called *L-measure space* if:
- X is an extremally disconnected compact Hausdorff space,
 - Ω is the Borel σ -algebra of X ,
 - μ is a finite measure such that $\mu(1) = 1$ and the null sets coincide with the meagre sets.
- (E. De Jonge, A.C.M. Van Rooij, Introduction to Riesz spaces, 1977)

A set $D \subseteq X$ is *meagre* if there exists a sequence $\{D_n\}_n$ of subsets of X such that $D \subset \bigcup_n D_n$ and every D_n is closed and has empty interior. A topological space is extremally disconnected if any open subset has a clopen closure.

Theorem.

The following categories are dually equivalent:

StCoRMV

- Objects: (A, s) state complete Riesz MV-algebra,
- Morphisms: $h: (A, s) \rightarrow (C, p)$ state-preserving Riesz MV-algebra homomorphism.

LMeasureSp

- Objects: (X, Ω, μ) L-measure space,
- Morphisms: a morphism from (X, Ω, μ) to (Y, Σ, ν) is a measure-preserving transformation, i.e. $T: X \rightarrow Y$ such that $T^{-1}(F) \in \Omega$ and $\mu(T^{-1}(F)) = \nu(F)$ for any $F \in \Sigma$.

$\mathcal{L}: \mathbf{LMeasureSp} \rightarrow \mathbf{StCoRMV}$

- ▶ $\mathcal{L}(X, \Omega, \mu) = (L^1(\mu)_u, s)$, where
 $s(f) = \int_X f \, d\mu$ for any $f \in L^1(\mu)_u$
- ▶ If $(X, \Omega, \mu), (Y, \Sigma, \nu) \in |\mathbf{LMeasureSp}|$ and
 $T: X \rightarrow Y$ is a measure-preserving transformation then let
 $U_T: L^1(\nu) \rightarrow L^1(\mu)$ be the induced operator
($U_T(g) = g \circ T$ for any $g \in L^1(\nu)$).
Hence U_T is an isometrical Riesz homomorphism.
We define $\mathcal{L}(T): L^1(\nu)_u \rightarrow L^1(\mu)_u$ by
 $\mathcal{L}(T)(f) = U_T(f) = f \circ T$ for any $f \in L^1(\nu)_u$.

$\mathcal{M}: \mathbf{StCoRMV} \rightarrow \mathbf{LMeasureSp}$

- ▶ $\mathcal{M}(A, s) = (X, \Omega, \mu)$, where $X = \text{Max}(A)$, Ω is the σ -algebra of Borel sets of X and μ is the unique regular measure corresponding to s .
- ▶ If $(A, s), (B, p) \in | \mathbf{StCoRMV} |$ and $h: A \rightarrow B$ is a state-preserving Riesz MV-algebra hom., $\mathcal{M}(h): \text{Max}(B) \rightarrow \text{Max}(A)$ by $\mathcal{M}(h)(\mathfrak{m}) = h^{-1}(\mathfrak{m})$ for any $\mathfrak{m} \in \text{Max}(B)$.

- ▶ MV-algebras and rings
- ▶ MV-algebras and Riesz spaces

- ▶ MV-algebras and rings
- ▶ MV-algebras and Riesz spaces
- ▶ Tensor products

The Tensor Product

Definition

If A, B, C are MV-algebras then $\beta : A \times B \rightarrow C$ is a bilinear function if it is linear in each argument.

We say that β is a bimorphism if it is bilinear and

$$\beta(a, b_1 \vee b_2) = \beta(a, b_1) \vee \beta(a, b_2),$$

$$\beta(a_1 \vee a_2, b) = \beta(a_1, b) \vee \beta(a_2, b),$$

$$\beta(a, b_1 \wedge b_2) = \beta(a, b_1) \wedge \beta(a, b_2),$$

$$\beta(a_1 \wedge a_2, b) = \beta(a_1, b) \wedge \beta(a_2, b).$$

Tensor product of MV-algebras (Mundici, 1999)

The tensor product of A and B is an MV-algebra $A \otimes B$ together with a bimorphism $\beta_{A,B} : A \times B \rightarrow A \otimes B$ satisfying the following universal property:

for any MV-algebra C and bimorphism $\lambda : A \times B \rightarrow C$ there exists an unique morphism $f : A \otimes B \rightarrow C$ such that $f \circ \beta_{A,B} \rightarrow \lambda$.

- ▶ We denote $a \otimes b := \beta_{A,B}(a, b)$ for any $a \in A$, $b \in B$. The MV-algebra $A \otimes B$ is generated by $\beta_{A,B}(A \times B) = \{a \otimes b \mid a \in A, b \in B\}$.
- ▶ $\iota_A : A \rightarrow A \otimes B$, $\iota_B : B \rightarrow A \otimes B$,
 $\iota_A(a) := a \otimes 1$, $\iota_B(b) := 1 \otimes b$
 ι_A and ι_B are embeddings (Montagna, Flaminio, private communication)

- ▶ Scalar extension property (Flondor, L., 2003)
If A is an MV-algebra then $[0, 1] \otimes A$ is a Riesz MV-algebra:
 $r(q \otimes a) := (rq) \otimes a$ for any $a \in A$, $r, q \in [0, 1]$
- ▶ An MV-algebra A is a Riesz MV-algebra iff $A \simeq [0, 1] \otimes A$.
- ▶ (Montagna, Flaminio, 2009) If A is an MV-algebra and $s : A \rightarrow [0, 1]$ is a state, then $(T = [0, 1] \otimes A, \sigma)$ is an SMV-algebra, where $\sigma : T \rightarrow T$ satisfies
 $\sigma(r \otimes a) = (rs(a)) \otimes 1$ for any $r \in [0, 1]$ and $a \in A$.

Categorical setting

- ▶ There is an adjunction between the category of MV-algebras and the category of Riesz MV-algebras.

$$\mathcal{R} : \mathbf{MV} \rightarrow \mathbf{RieszMV}, \mathcal{R}(A) := [0, 1] \otimes A$$

$\mathcal{M}\mathcal{V} : \mathbf{RieszMV} \rightarrow \mathbf{MV}(\text{forgetful})$, \mathcal{R} is the left adjoint of $\mathcal{M}\mathcal{V}$.

- ▶ If A is an MV-algebra then $[0, 1] \otimes A$ is the free Riesz MV-algebra over A .
- ▶ $[0, 1]_{\mathbf{Q}} \otimes A$ is the divisible hull of A .

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Do we have a similar approach for PMV-algebras?

Problem (IL)

Given an MV-algebra A , our goal is to define the free PMV-algebra over A .

PMV-algebra

(P, \cdot) , P -MV-algebra, $\cdot: P \times P \times P$:

1. $z \cdot (x \oplus y) = (z \cdot x) \oplus (z \cdot y)$ whenever $x \odot y = 0$,
 2. $(x \oplus y) \cdot z = (x \cdot z) \oplus (y \cdot z)$ whenever $x \odot y = 0$,
 3. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,
 4. $1 \cdot x = x \cdot 1 = x$,
- for any $x, y, z \in P$.

- ▶ A MV-algebra, (P, \cdot) PMV-algebra, $g : A \rightarrow P$ MV-morphism $\lambda : A \times A \rightarrow P$, $\lambda(a, b) := g(a) \cdot g(b)$ for any $a, b \in A$ there exists an unique MV-morphism $\tilde{g} : A \otimes A \rightarrow P$ such that $\tilde{g}(a \otimes b) = \lambda(a, b) = g(a) \cdot g(b)$ for any $a, b \in A$.

Note that $\tilde{g}(1 \otimes a) = \tilde{g}(a \otimes 1)$ for any morphisms $g : A \rightarrow P$.

- ▶ We want to define the product operation using \otimes , so we would like to have $1 \otimes a = a \otimes 1$. In order to obtain this, we will slightly modify Mundici's definition for \otimes .

- ▶ A MV-algebra, (P, \cdot) PMV-algebra, $g : A \rightarrow P$ MV-morphism $\lambda : A \times A \rightarrow P$, $\lambda(a, b) := g(a) \cdot g(b)$ for any $a, b \in A$ there exists an unique MV-morphism $\tilde{g} : A \otimes A \rightarrow P$ such that $\tilde{g}(a \otimes b) = \lambda(a, b) = g(a) \cdot g(b)$ for any $a, b \in A$.

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- ▶ We want to define the product operation using \otimes , so we would like to have $1 \otimes a = a \otimes 1$. In order to obtain this, we will slightly modify Mundici's definition for \otimes .
- ▶ If A and B are two MV-algebras, the tensor product is defined by $A \times B = \text{Free}_{MV}(\text{Set}(A \times B))/I$, where the ideal I is chosen such that the universal map $\beta_{A,B} : A \times B \rightarrow A \otimes B$ has the desired properties. Mundici defined I such that β is a bimorphism.

Let A be an MV-algebra and $n \geq 2$. We set

$T^n(A) := \text{Free}_{MV}(\text{Set}(A^n))/I$, where I is the ideal generated by:

- ▶ $d((0, \dots, 0), 0), d((1, \dots, 1), 1),$
- ▶ $d((a_1, \dots, a_i \vee b_i, \dots, a_n),$
 $(a_1, \dots, a_i, \dots, a_n) \vee (a_1, \dots, b_i, \dots, a_n))$
- ▶ $d((a_1, \dots, a_i \wedge b_i, \dots, a_n),$
 $(a_1, \dots, a_i, \dots, a_n) \wedge (a_1, \dots, b_i, \dots, a_n))$
- ▶ $d((a_1, \dots, a_i, \dots, a_n) \odot (a_1, \dots, b_i, \dots, a_n), 0)$
 whenever $a_i \odot b_i = 0,$
- ▶ $d((a_1, \dots, a_i \oplus b_i, \dots, a_n),$
 $(a_1, \dots, a_i, \dots, a_n) \oplus (a_1, \dots, b_i, \dots, a_n))$
 whenever $a_i \odot b_i = 0$
- ▶ $d((a_1, \dots, a_i, 1, \dots, 1), (b_1, \dots, b_n))$ whenever there exist
 $k_1 \leq \dots \leq k_i \in 1..n$ such that $b_{k_1} = a_1, \dots, b_{k_i} = a_i$ and
 $b_k = 1$ otherwise,
 for any $i \in 1..n.$

Recall $d(a, b) = (a \otimes b^*) \oplus (b \otimes a^*).$

- ▶ Denote $a_1 \otimes \cdots \otimes a_n \in T^n(A)$ the equivalence class of (a_1, \dots, a_n) .

The following properties hold:

- ▶ For any n , the function $\beta_n : A^n \rightarrow T^n(A)$, $(a_1, \dots, a_n) \mapsto a_1 \otimes \cdots \otimes a_n$ is multilinear. Moreover, fixing $n - 1$ components we get lattice homomorphisms.

Universal property of $T^n(A)$

For any MV-algebra M and for any multilinear function $\lambda : A^n \rightarrow M$, there exists an unique MV-algebra homomorphism $\tilde{f} : T^n(A) \rightarrow M$ such that $\tilde{f} \circ \beta_k = \lambda$, i.e. $\tilde{f}(a_1 \otimes \cdots \otimes a_n) = \lambda(a_1, \dots, a_n)$ for any $(a_1, \dots, a_n) \in A^n$.

The tensor algebra

$$T^1(A) := A, T^2(A), \dots, T^n(A), \dots$$

- ▶ whenever $k \leq n$ let $\epsilon_{kn} : T^k(A) \rightarrow T^n(A)$ be the unique MV-algebra homomorphism with

$$\epsilon_{kn}(a_1 \otimes \cdots \otimes a_k) = a_1 \otimes \cdots \otimes a_k \otimes 1 \otimes \cdots \otimes 1$$

for any $a_1, \dots, a_k \in A$.

- ▶ $(\{T^n(A)\}_{n \geq 1}, \{\epsilon_{kn}\}_{k \leq n})$ is a direct system and let $(T(A), \{g_n\}_{n \geq 1})$ be its direct limit.

We call $T(A)$ **the tensor algebra** of A .

- ▶ $g_1 : A \rightarrow T(A)$ is an embedding.

Theorem

The tensor algebra $T(A)$ is a PMV-algebra for any MV-algebra A .

Consequences

- ▶ $T(A)$ is the free PMV-algebra generated by A . For any PMV-algebra P and for any MV-algebra morphism $f : A \rightarrow P$, there exists an unique PMV-algebra morphism \tilde{f} such that $\tilde{f} \circ g_1 = f$.

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- ▶ $T(A)$ is the free PMV-algebra generated by A . For any PMV-algebra P and for any MV-algebra morphism $f : A \rightarrow P$, there exists a unique PMV-algebra morphism \tilde{f} such that $\tilde{f} \circ g_1 = f$.
- ▶ There is an adjunction between the category of MV-algebras and the category of PMV-algebras.
 $\mathcal{T} : \mathbf{MV} \rightarrow \mathbf{PMV}$, $\mathcal{T}(A) := T(A)$,
 $\mathcal{MV} : \mathbf{PMV} \rightarrow \mathbf{MV}$ (forgetful), \mathcal{T} is the left adjoint of \mathcal{MV} .
- ▶ If A is the free MV-algebra generated by X , then $T(A)$ is the free PMV-algebra generated by X .
- ▶ An MV-algebra A is a PMV-algebra iff A is a retract of $T(A)$.

Theorem

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Proof:

- ▶ Recall that $(T(A), \{g_n\}_{n \geq 1})$ be its direct limit of $(\{T^n(A)\}_{n \geq 1}, \{\epsilon_{kn}\}_{k \leq n})$.
- ▶ Intuitively, the product should behave in the following way:
 $(a_1 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes \cdots \otimes b_m) = a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m$.

▶ Claim:

For any n, m there exists a unique bilinear function

$\pi_{n,m} : T^n(A) \times T^m(A) \rightarrow T^{n+m}(A)$ such that

$$\pi_{n,m}(a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_m) = a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m.$$

- ▶ If $\mathbf{x}, \mathbf{y} \in T(A)$ then there exist $n, m, \mathbf{a} \in T^n(A), \mathbf{b} \in T^m(A)$ such that $\mathbf{x} = g_n(\mathbf{a})$ and $\mathbf{y} = g_m(\mathbf{b})$. Then we define

$$\mathbf{x} \cdot \mathbf{y} := g_{n+m}(\pi_{n,m}(\mathbf{a}, \mathbf{b})).$$

Claim:

For any n, m there exists a unique bilinear function

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Proof:

- ▶ Let $a_1, \dots, a_n \in A$ and define $\lambda : A^m \rightarrow T^{n+m}(A)$ by $\lambda(b_1, \dots, b_m) = a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m$.
Since λ is a multilinear map, there exists a unique MV-algebra homomorphism $F_{a_1, \dots, a_n} : T^m(A) \rightarrow T^{n+m}(A)$ such that $F_{a_1, \dots, a_n} \circ \beta_m = \lambda$.
- ▶ Let $\mathbf{b} \in T^m(A)$ and define $\gamma : A^n \rightarrow T^{n+m}(A)$ by $\gamma(a_1, \dots, a_n) = F_{a_1, \dots, a_n}(\mathbf{b})$.
Since γ is a multilinear map, there exists a unique MV-algebra homomorphism $G_{\mathbf{b}} : T^n(A) \rightarrow T^{n+m}(A)$ such that $G_{\mathbf{b}} \circ \beta_n = \gamma$, i.e. $G_{\mathbf{b}}(a_1 \otimes \cdots \otimes a_n) = F_{a_1, \dots, a_n}(\mathbf{b})$.

Proof of the claim:

- ▶ For any $\mathbf{b} \in T^m(A)$ we have defined $G_{\mathbf{b}} : T^n(A) \rightarrow T^{n+m}(A)$ MV-algebra homomorphism.
- ▶ If $\mathbf{a} \in T^n(A)$ and $\mathbf{b} \in T^m(A)$ then we define $\pi_{n,m}(\mathbf{a}, \mathbf{b}) := G_{\mathbf{b}}(\mathbf{a})$.
- ▶ $\pi_{n,m}$ is linear:
 - $G_{\mathbf{b}}(\mathbf{a} + \mathbf{c}) = G_{\mathbf{b}}(\mathbf{a}) + G_{\mathbf{b}}(\mathbf{c})$
 - $G_{\mathbf{b}+\mathbf{c}} = G_{\mathbf{b}} + G_{\mathbf{c}}$ (they are equal on generators)

Thank you for your attention!