Non-safe structures in fuzzy logics and game semantics

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Outline

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Games in fuzzy logic – history

Dialogue (proof-theoretical) games

K. Lorenzen (1953) - dialogue games for intuitionist/classical logic
R. Giles (1974/77) - dialogue style game for Lukasiewicz logic
Ch. Fermüller (2003, ...) - Giles game for Gödel, Product
A. Ciabattoni, G. Metcalfe - (2005, ...)

Evaluation (model-theoretical) games

J. Hintikka, G. Sandu (1997) - evaluation game for classical logic
P. Cintula, O.M.(2006, 2009) - evaluation game for Lukasiewicz logic

D. Mundici (1993) - Rényi-Ulam game for finitely valued Lukasiewicz logic

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Lukasiewicz predicate logic-axiomatics

Propositional axioms

(Ł1) $\varphi \to (\psi \to \varphi)$

(Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(Ł3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

(Ł4) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

Predicate axioms

(∀1) $(\forall x)\varphi(x) \to \varphi(t)$, where t is substitutable

(∀2) $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, where x is not free in χ,

Inference rules

Modus Ponens, generalisation

Lukasiewicz predicate logic - semantics

Let **Γ** be a predicate language, **Ł** an MV-algebra, **M** an **Ł**-structure for **Γ**,v an **M**-evaluation.

$\|x\|_{\mathbf{M},v} = v(x)$ and $\|f(t_1,\ldots,d_n)\|_{\mathbf{M},v} = f_{\mathbf{M}}(\|t_1\|_{\mathbf{M},v},\ldots,\|t_n\|_{\mathbf{M},v})$.

A *truth value of the formula* φ in **M** for an evaluation v:

$$\begin{aligned}
\|P(t_1,\ t_2,\ \ldots,\ t_n)\|^{\mathbf{Ł}}_{\mathbf{M},v} &= P_{\mathbf{M}}(\|t_1\|_{\mathbf{M},v},\ \|t_2\|_{\mathbf{M},v},\ \ldots,\ \|t_n\|_{\mathbf{M},v})\,,\\
\|\varphi \oplus \psi\|^{\mathbf{Ł}}_{\mathbf{M},v} &= \|\varphi\|^{\mathbf{Ł}}_{\mathbf{M},v} \oplus \|\psi\|^{\mathbf{Ł}}_{\mathbf{M},v}\,,\\
\|\neg\varphi\|^{\mathbf{Ł}}_{\mathbf{M},v} &= \neg\|\varphi\|^{\mathbf{Ł}}_{\mathbf{M},v}\,,\\
\|0\|_{\mathbf{M},v} &= 0\,,\\
\|(\forall x)\varphi\|_{\mathbf{M},v} &= \inf\{\|\varphi\|^{\mathbf{Ł}}_{\mathbf{M},v'} \mid v' \equiv_x v\}\\
\|(\exists x)\varphi\|_{\mathbf{M},v} &= \sup\{\|\varphi\|^{\mathbf{Ł}}_{\mathbf{M},v'} \mid v' \equiv_x v\}
\end{aligned}$$

If infimum (supremum) does not exist, we take its value as undefined.

Non-safe structure-example

Example

L - the subalgebra of the standard MV-algebra with the domain $[0,1] \cap Q$.
q - an irrational number, $q \geq \frac{1}{2}$
a_i - a descending sequence of rationals, $lim(a_i) = q$.
predicate language with one unary predicate P
M - **L**-structure, the domain of $\mathbf{M} = \mathcal{N}$
interpretation of P: $P_{\mathbf{M}}(i) = a_i$.

- $\|\forall x P(x)\| = inf\{a_i | a_i \in \mathbf{L}\}$
- $inf\{a_i | a_i \in \mathbf{L}\} = q, q \notin \mathbf{L}$,
- $\|\forall x P(x)\|$ is undefined
- **M** is not a safe **L**-structure

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Classical evaluation games

Evaluation game for classical logic (J. Hintikka, G. Sandu, 1997)
Is a formula φ true in a model **M** w. r. to an **M**-evaluation v?

- a zero-sum game of two players (Eloise and Abelard)
- $\mathcal{E}$ is the initial Verifier ($||\varphi||_{\mathbf{M},v} = 1$),
  $\mathcal{A}$ is the initial Falsifier ($||\varphi||_{\mathbf{M},v} = 0$)
- rules of a game $G(\mathbf{M}, v, \varphi)$ are given by the structure of φ.

A formula φ is (game-theoretically) true in $(\mathbf{M}, v)$ iff there exits a winning strategy for the initial verifier in the evaluation game $G(\mathbf{M}, v, \varphi)$.

Theorem (Determinedness)

Classical evaluation games are determined (either Eloise or Abelard has a winning strategy).

Theorem (Correspondence)

Game theoretical truth corresponds to the Tarskian truth:
$\mathbf{M}, v, \models_{GTS} \varphi$ *iff* $\mathbf{M}, v, \models \varphi$

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Fuzzy evaluation games

Evaluation game for fuzzy logics (P. Cintula, O. M. 2006, 2009)
Is a formula φ true in an **L**-structure **M** w. r. to an **M**-evaluation v true *at least in the degree r*?

- extension of classical evaluation game (zero sum, two players, two roles),
- one more parameter $r \in \mathbf{L}$
- $\mathcal{E}$: $\|\varphi\|_{\mathbf{M},v} \geq r$
- $\mathcal{A}$: $\|\varphi\|_{\mathbf{M},v} < r$

Definition (Lukasiewicz evaluation game)

Let **L** be an MV-chain, **M** be an **L**-structure, φ a formula, v an **M**-valuation, and $r \in L$. The fuzzy evaluation game $G^{\mathbf{L}}_{M}(\varphi, v, r)$ has the following moves:

Lukasiewicz evaluation game

Terminating rules

(*at*) (ψ, v, r), where ψ is an atomic formula:
the end of the game,
if $\|\psi\|_{\mathbf{M},v} \geq r$ (the current) $\mathcal{V}$ wins,
otherwise $\mathcal{F}$ wins.

(0) $(\varphi, v, 0)$:
the end of the game, the current $\mathcal{V}$ wins.

Negation

(¬) $(\neg\psi, v, r)$:
$\mathcal{F}$ chooses $\varepsilon, r \geq \varepsilon > 0$,
role switch, game continues as $(\psi, v, (1 - r) + \varepsilon)$

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Lukasiewicz evaluation game

Disjunction

($\oplus$) $(\psi_1 \oplus \psi_2, v, r)$:
$\mathcal{V}$ chooses $r' \leq r$,
$\mathcal{F}$ chooses whether to play (ψ_1, v, r') or $(\psi_2, v, r - r')$.

($\vee$) $(\psi_1 \vee \psi_2, v, r)$:
$\mathcal{V}$ chooses whether to play (ψ_1, v, r) or (ψ_2, v, r).

Lukasiewicz evaluation game

Conjunction

($\otimes$) $(\psi_1 \otimes \psi_2, v, r)$:
$\mathcal{V}$ chooses $r' \leq 1 - r$,
$\mathcal{F}$ chooses whether to play $(\psi_1, v, r + r')$ or $(\psi_2, v, r + (1 - r - r'))$.

($\wedge$) $(\psi_1 \wedge \psi_2, v, r)$:
$\mathcal{F}$ chooses whether to play (ψ_1, v, r) or (ψ_2, v, r).

Lukasiewicz evaluation game

General quantifier$((\forall x)\psi, v, r)$:
$\mathcal{V}$ claims that $\inf(||\psi||_{v[x]}) \geq r$
$\mathcal{F}$ has to provide a counterexample - an a' such that $(||\psi||_{v[x:a']} < r)$

($\forall$) $((\forall x)\psi, v, r)$:
$\mathcal{F}$ chooses $a \in M$,
game continues as $(\psi, v[x : a], r)$.

(Non)existence of the infimum or of an witnessing element does not influence Falsifier's choice.

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Lukasiewicz evaluation game

Existential quantifier $((\exists x)\psi, v, r)$:
$\mathcal{V}$ claims that $\sup(\|\psi\|_{v[x]}) \geq r$
$\mathcal{V}$ has to provide a witness - an a' such that $(\|\psi\|_{v[x:a']} \geq r)$

In witnessed models:

$(\exists')$ $((\exists x)\psi, v, r)$:
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Lukasiewicz evaluation game

Existential quantifier (cnt.)
If the supremum is proper, i.e., $\|\psi\|_{v[x:a']} < \sup(\|\psi\|_{v[x]})$ for all $a' \in M$, $\mathcal{V}$ cannot never provide a witness, she would always lose.
To make the game fair we weaken the classical rule:

- $\mathcal{F}$ decreases r (it is in his interest to decrease it as little as possible)
- $\mathcal{V}$ finds an element in the domain to witness the weakened condition.

($\exists$) $((\exists x)\psi, v, r)$:
$\mathcal{F}$ chooses $\varepsilon, 0 < \varepsilon \leq r$
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Correspondence theorem

Theorem (Determinedness)

Fuzzy evaluation games are determined—either Eloise or Abelard has a winning strategy for every $(\varphi, \mathbf{M}, v, r)$.

Moreover, the game-theoretical value (existence of winning strategies for Eloise for a certain r) coincides in the case of safe structures with the standard Tarskian value.

Theorem (Correspondence - Cintula, M.)

Let $\mathbf{L}$ *be an* MV*-chain,* $\mathbf{M}$ *be a safe* $\mathbf{L}$*-structure,* φ *a formula,* v *an* $\mathbf{M}$*-valuation, and* $r \in L$*. Then Eloise has a winning strategy for the* $(\mathbf{M}, \mathbf{L})$*-Game* (φ, v, r) *iff* $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} \geq r$.

Powers of players

Powers of the players - ”what they can win”

Definition (Powers of players)

Let $\mathbf{L}$ be an MV-chain, $\mathbf{M}$ be an $\mathbf{L}$-structure, φ a formula, and v an $\mathbf{M}$-valuation. We define:

$\mathcal{E}(\varphi) =_{\mathrm{df}} \{ r \mid$ Eloise has a winning strategy for the game $G^{\mathbf{L}}_{\mathbf{M}}(\varphi, v, r)\}$.
$\mathcal{A}(\varphi) =_{\mathrm{df}} \{ r \mid$ for any $r' \in \mathbf{L}$, $r' > r$, Abelard has a w. s. for $G^{\mathbf{L}}_{\mathbf{M}}(\varphi, v, r')\}$.

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Properties of powers

Let **L** be an MV-chain, **M** be an **L**-structure, φ a formula, and v an **M**-valuation. Then:

(i)	$0_L \in \mathcal{E}(\varphi)$	(ii)	$1_L \in \mathcal{A}(\varphi)$;
(iii)	$\mathcal{E}(\varphi)$ is a lower set;	(iv)	$\mathcal{A}(\varphi)$ is an upper set;
(v)	$\mathcal{A}(\varphi) \cup \mathcal{E}(\varphi) = L$;	(vi)	$\lVert\mathcal{A}(\varphi) \cap \mathcal{E}(\varphi)\rVert \leq 1$;

(vii) For a *safe* **M**: $\mathcal{A}(\mathbf{M}, \mathbf{L}, v, \varphi) \cap \mathcal{E}(\mathbf{M}, \mathbf{L}, v, \varphi) = \{\lVert\varphi\rVert^{\mathbf{L}}_{\mathbf{M},v}\}$.

Lemma (Characterisation of safe structures)

Let **L**, **M**, v, φ *be as before. The structure* **M** *is safe iff the powers of players have a nonepmty intersection for any formula* φ *and valuation* v.

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$$\mathcal{E}(\varphi \oplus \psi) = \{r \oplus s \mid r \in \mathcal{E}(\varphi) \text{ and } s \in \mathcal{E}(\psi)\}; \quad \mathcal{A}(\varphi \oplus \psi) = \mathcal{A}(\varphi) \oplus \mathcal{A}(\psi);$$
$$\mathcal{E}(\varphi \otimes \psi) = \mathcal{E}(\varphi) \otimes \mathcal{E}(\psi) \quad \mathcal{A}(\varphi \otimes \psi) = \mathcal{A}(\varphi) \otimes \mathcal{A}(\psi);$$

$$\mathcal{E}(\varphi \vee \psi) = \mathcal{E}(\varphi) \cup \mathcal{E}(\psi); \quad \mathcal{A}(\varphi \vee \psi) = \mathcal{A}(\varphi) \cap \mathcal{A}(\psi);$$

$$\mathcal{E}(\varphi \wedge \psi) = \mathcal{E}(\varphi) \cap \mathcal{E}(\psi); \quad \mathcal{A}(\varphi \wedge \psi) = \mathcal{A}(\varphi) \cup \mathcal{A}(\psi);$$

$$\mathcal{E}(\neg\varphi) = \{\neg r \mid r \in \mathcal{A}(\varphi)\} \quad \mathcal{A}(\neg\varphi) = \{\neg r \mid r \in \mathcal{E}(\varphi)\};$$

$$\mathcal{E}((\forall x)\varphi) = \bigcap_{a \in M} \mathcal{E}(v[x = a], \varphi); \quad \mathcal{A}((\forall x)\varphi) = \mathrm{Cl}(\bigcup_{a \in M} \mathcal{A}(v[x = a], \varphi));$$

$$\mathcal{E}((\exists x)\varphi) = \mathrm{Cl} \bigcup_{a \in M} \mathcal{E}(v[x = a], \varphi); \quad \mathcal{A}((\exists x)\varphi) = \bigcup_{a \in M} \mathcal{A}(v[x = a], \varphi).$$

$\mathrm{Cl}(A) = A \cup inf(A) \cup sup(A)$, if $sup(A), inf(A)$ exist
$\mathrm{Cl}(A) = A$ otherwise

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$||\forall x P(x)|| = inf\{a_i | a_i \in \mathbf{L}\}$ undefined, but

- $\mathcal{E}(\forall x P(x)) = [0, q)$.
- $\varphi = (\forall x)P(x) \oplus (\forall x)P(x)$ is also undefined.
- but $\mathcal{E}(\varphi) = \mathbf{L}$ and $\mathcal{A}(\varphi) = \{1\}$.

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G-truth

A formula φ is (fully) true in the game-theoretical semantics iff $\mathcal{E}(\varphi) = \mathbf{L}$ (Eloise has winning strategy for any value from **L**) iff ($1_L \in \mathcal{E}(\varphi)$) (Eloise has a winning strategy for the value 1_L)

Definition (G-truth)

Let **L** be an MV-chain, **M** be an *arbitrary* **L**-structure and v an **M**-valuation. Then we say that

- φ is true in $(\mathbf{M}, \mathbf{L})$ for a valuation v with respect to the game-theoretical semantics (G-true): $(\mathbf{M}, \mathbf{L})_v \models_G \varphi$ iff $\mathcal{E}(\mathbf{M}, \mathbf{L}, v, \varphi) = \mathbf{L}$.
- φ is G-true in the **L**-structure **M**: $(\mathbf{M}, \mathbf{L}) \models_G \varphi$ iff $\mathcal{E}(\mathbf{M}, \mathbf{L}, v, \varphi) = \mathbf{L}$ for any v
- φ is a G-tautology: $\models_G \varphi$ iff φ is G-true in all **L**-structures **M**.

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Theorem (Completeness for non-safe structures)

Let $\mathbf{\Gamma}$ *be a predicate language and* φ *a formula. Then the following are equivalent:*

(i) $\vdash \varphi$.

(ii) $(\mathbf{M}, \mathbf{L}) \models_G \varphi$ *for every* MV*-chain* $\mathbf{L}$ *and every* $\mathbf{L}$*-structure* $\mathbf{M}$.

Proof

(ii) $\rightarrow$ (i)

as in safe models

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to be proven

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G-completeness - proof

We denote $\mathcal{E}(\mathbf{M}, v, \varphi) = ||\varphi||_{\mathcal{E}}^{\mathbf{M},v}$

Lemma

(i) *for* $r \in 1_{\mathbf{L}}, 0_{\mathbf{L}} < r < 1_{\mathbf{L}}$ *it holds* $r \in ||\varphi||_{\mathcal{E}}$ *iff* $1_{\mathbf{L}} \ominus r \in ||\neg\varphi||_{\mathcal{E}}$

(ii) *A formula* $\varphi \to \psi$ *is G-tautology iff* $||\varphi||_{\mathcal{E}}^{\mathbf{M},v} \subseteq ||\psi||_{\mathcal{E}}^{\mathbf{M},v}$ *for any* $\mathbf{M}, v$

Proof.

(i) from the definition of the $\neg$-move

(ii)$1_{\mathbf{L}} \in ||\neg\varphi \oplus \psi||_{\mathcal{E}}^{\mathbf{M},v}$ iff there are r_1, r_2 such that $1_{\mathbf{L}} \ominus r_1 \in ||\neg\varphi||_{\mathcal{E}}^{\mathbf{M},v}, r_2 \in ||\psi||_{\mathcal{E}}^{\mathbf{M},v}$ and $1_{\mathbf{L}} \ominus r_1 \oplus r_2 \geq 1_{\mathbf{L}}$ iff there are $r_1 \in ||\varphi||_{\mathcal{E}}^{\mathbf{M},v}, r_2 \in ||\psi||_{\mathcal{E}}^{\mathbf{M},v}$ and $r_1 \geq r_2$ in fact for any $r_1 \in ||\varphi||_{\mathcal{E}}^{\mathbf{M},v}$ there must be $r_2 \in ||\psi||_{\mathcal{E}}^{\mathbf{M},v}$ s.t. $r_1 \geq r_2$ (otherwise Eloise cannot win the corresponding move for the value $1_{\mathbf{L}}$ and finally from $||\ ||_{\mathcal{E}}$ being a lower set $||\varphi||_{\mathcal{E}} \subseteq ||\psi||_{\mathcal{E}}$

□

G-truth - correctness

Propositional axioms

$\varphi \to (\psi \to \varphi)$

if $||\varphi||_\mathcal{E} = 1_\mathbf{L}$ then $||\psi||_\mathcal{E} \subseteq ||\varphi||_\mathcal{E}$ hence $||\psi \to \varphi||_\mathcal{E} = 1_\mathbf{L}$ as well and $||\varphi||_\mathcal{E} \subseteq ||\psi \to \varphi||_\mathcal{E}$

if $||\varphi||_\mathcal{E} \subset 1_\mathbf{L}$, then from the definition of the $\oplus$-move there have to be r_1, r_2, r_3, such that

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we take $r_3 = 1_\mathbf{L} \ominus r_1$

...

Predicate axioms

$\forall x\varphi(x) \to \varphi(t)$

$r \in ||\forall x\varphi(x)||^v_\mathcal{E}$ then $r \leq inf||\varphi(x)||_{v(x:a)} \leq \varphi(t)$, so $||\forall x\varphi(x)||^v_\mathcal{E} \subseteq ||\varphi(t)||^v_\mathcal{E}$ for all v.

Inference rules ...

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