Some Categorical Equivalences Involving Gödel Algebras

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Gödel Algebras

- A Gödel algebra $\mathbf{A} = \langle A, \sqcap, \sqcup, *, \rightarrow, \bot, \top \rangle$ is an integral commutative bounded residuated lattice that is
  - Idempotent: $x * x = x$ (and so $x \sqcap y = x * y$);
  - Prelinear: $(x \rightarrow y) \sqcup (y \rightarrow x) = \top$.

- The variety of Gödel algebras is generated by the class of chains where

$$
\begin{array}{rcl}
x \sqcap y & = & \min(x, y) \\
x \sqcup y & = & \max(x, y) \\
x \rightarrow y & = & \begin{cases} \top & x \leq y \\ y & x > y \end{cases}
\end{array}
$$

- In particular the variety is generated by the algebra over $[0, 1]$

Nilpotent Minimum Algebras

- A Nilpotent Minimum algebra [Esteva, Godo] $\mathbf{A} = \langle A, \sqcap, \sqcup, *, \to, \bot, \top \rangle$ is a prelinear integral commutative bounded residuated lattice that satisfies
  - $((x * y) \to \bot) \sqcup ((x * y) \to (x \sqcap y)) = \top$.

- Given any bounded chain C and any involutive negation n over C we can define an NM chain where

$$
\begin{aligned}
x * y &= \begin{cases} \bot & x \leq n(y) \\ \min(x, y) & \text{oth.} \end{cases} \\
x \to y &= \begin{cases} \top & x \leq y \\ \max(n(x), y) & \text{oth.} \end{cases}
\end{aligned}
$$

  so that $n(x) = x \to \bot$.

- The variety is generated by the class of NM chains, and in particular by the algebra over $[0, 1]$ where $n(x) = 1 - x$.

IUML Algebras

- An IUML algebra [Metcalfe, Montagna] $\mathbf{A} = \langle A, \sqcap, \sqcup, *, \to, \bot, \top, \mathbf{e} \rangle$ is an idempotent commutative bounded residuated lattice that satisfies
  - $\mathbf{e} \leq (x \to y) \sqcup (y \to x)$;
  - $(x \to \mathbf{e}) \to \mathbf{e} = x$

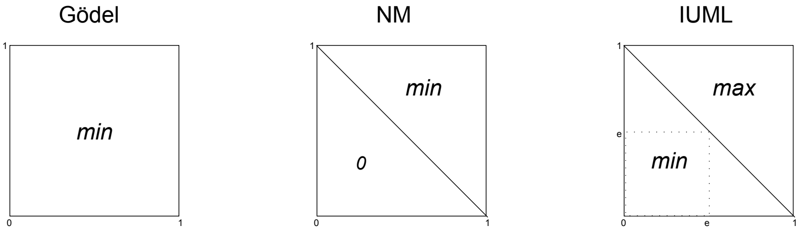
- Given any bounded chain C and any involutive negation n (with a fixed point $\mathbf{e}$) over C we can define a IUML chain where

$$
\begin{aligned}
x * y &= \begin{cases} \min(x, y) & x \leq n(y) \\ \max(x, y) & \text{oth.} \end{cases} \\
x \to y &= \begin{cases} \max(n(x), y) & x \leq y \\ \min(n(x), y) & \text{oth.} \end{cases}
\end{aligned}
$$

  so that $n(x) = x \to \mathbf{e}$.

- The variety is generated by the class of IUML chains, and in particular by the algebra over $[0, 1]$ where $n(x) = 1 - x$.

Gödel, NM, and IUML: the connection



- We use the fact that every finite (Gödel) algebra is isomorphic to a direct product of directly indecomposable (Gödel) algebras.
- Given a dir. ind. Gödel algebra, take its hoop subreduct $\langle A, \sqcap, \sqcup, \to, \top \rangle$.
- For $x \in A$ there is a set of meet-irreducible elements I_x such that

$$x = \bigwedge \{m_i \mid m_i \in I_x\}$$

- Each I_x is non-redundant, i.e. if $m_i > m_j$, and $m_j \in I_x$, then $m_i \notin I_x$.

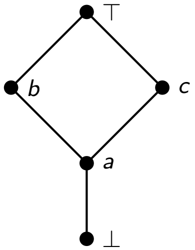
- We define a IUML algebra from A using the sets of meet-irreducible elements.
- Let

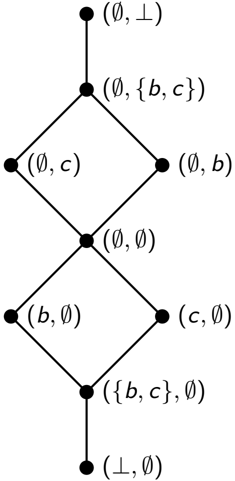
$$C^{-} = \{(\bot, \emptyset)\} \cup \{(I_x, \emptyset)\}_{x \in A} \cup \{(\emptyset, \emptyset)\}$$
$$C^{+} = \{(\emptyset, \bot)\} \cup \{(\emptyset, I_x)\}_{x \in A} \cup \{(\emptyset, \emptyset)\}$$

- Define the structure $C^{-} \cup C^{+}$ with the following order $\leq$:
  - $(E, \emptyset) \leq (\emptyset, F)$
  - $(E, \emptyset) \leq (F, \emptyset)$ iff $\bigwedge E \leq \bigwedge F$
  - $(\emptyset, E) \leq (\emptyset, F)$ iff $\bigwedge E \geq \bigwedge F$

Let

$$C^- = \{(\bot, \emptyset)\} \cup \{(I_x, \emptyset)\}_{x \in A} \cup \{(\emptyset, \emptyset)\}$$
$$C^+ = \{(\emptyset, \bot)\} \cup \{(\emptyset, I_x)\}_{x \in A} \cup \{(\emptyset, \emptyset)\}$$





From Gödel to IUML (III)

Let

$$C^- = \{(\bot, \emptyset)\} \cup \{(I_x, \emptyset)\}_{x \in A} \cup \{(\emptyset, \emptyset)\}$$
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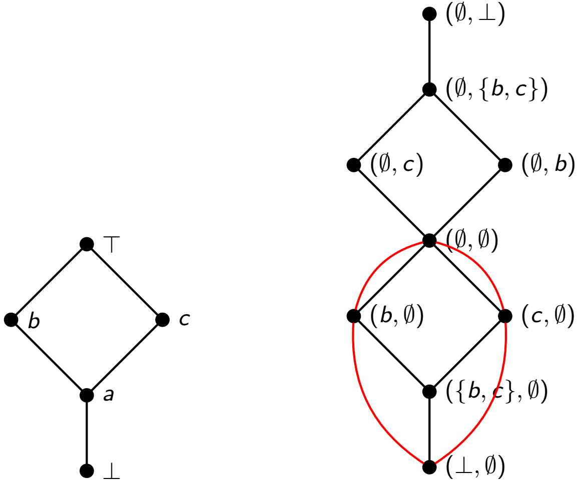

Let

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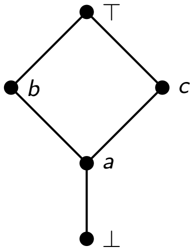
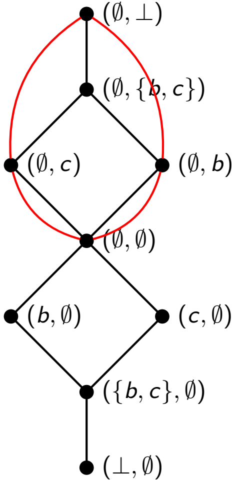
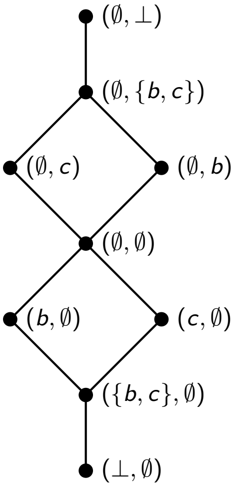




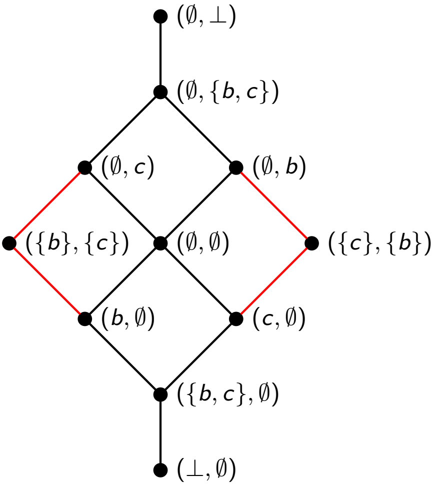
From Gödel to IUML (IV)

- Define the operation $\odot$ over $C^- \cup C^+$ as follows
$$(E, F) \odot (G, H) = (E \cup G, F \cup H).$$
- $C^- \cup C^+$ is not closed under $\odot$, so let $C^\star$ the corresponding closure.
- The elements of $C^\star$ are assumed to be non-redundant, i.e. for any (E, F), whenever $m_i \in E$, $m_j \in F$ and $m_i \leq m_j$, then (E, F) is reduced to $(E, F \backslash \{m_j\})$.
- $\neg(E, F) := (F, E)$, and $(E, F) \to (G, H) := \neg((E, F) \odot \neg(G, H))$.
- $\langle C^\star, \sqcap, \sqcup, \odot, \to, (\bot, \emptyset), (\emptyset, \bot), (\emptyset, \emptyset)\rangle$ is an IUML-algebra, and it is directly indecomposable.
- It can be easily seen that this construction is functorial from the category of directly indecomposable finite Gödel algebras to the category of directly indecomposable finite IUML-algebras.

From Gödel to IUML (V)



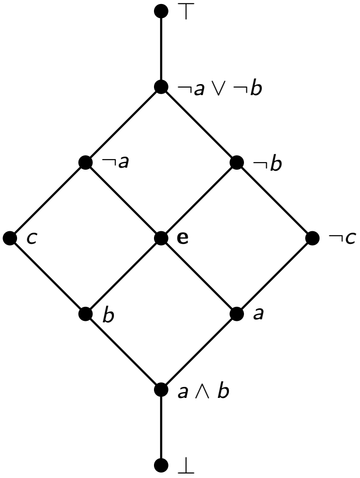
From Gödel to IUML (V)



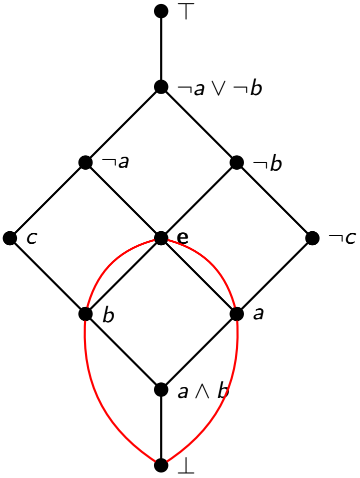
From IUML to Gödel (I)

- Take an IUML-algebra $\mathbf{B}$, and let $X = \{x \mid x \leq \mathbf{e}\}$.
- Let $\sqcap^{\mathbf{e}}$, $\rightarrow^{\mathbf{e}}$ be the restriction of the corresponding operations of $\mathbf{B}$ over X.
- Then $\langle X, \sqcap^{\mathbf{e}}, \rightarrow^{\mathbf{e}}, \bot, \mathbf{e}\rangle$ is a directly indecomposable Gödel algebra.
- It can be easily seen that this construction is functorial from the category of directly indecomposable finite IUML-algebras to the category of directly indecomposable finite Gödel algebras.

From IUML to Gödel (II)



From IUML to Gödel (II)



Theorem

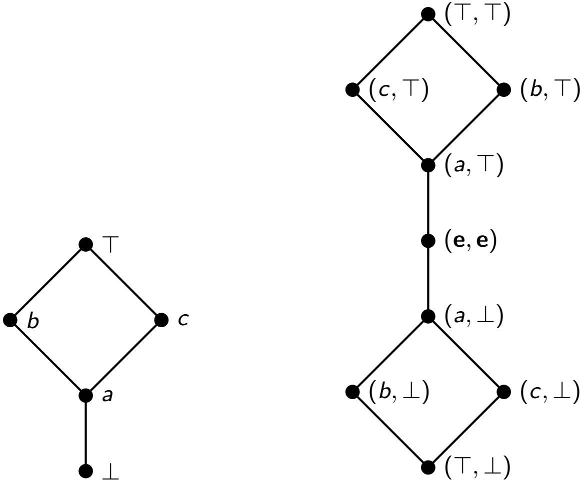
The category of finite Gödel algebras with homomorphisms, is equivalent to the category of finite IUML algebras with homomorphisms.

- Again, we work with directly indecomposable algebras.
- All the results were given (in a more general setting) by M. Busaniche.
- Take the variety $\mathbb{NM}^+$ of NM algebras with a fixed point **e**, axiomatized as an expansion of $\mathbb{NM}$ with the equation $\mathbf{e} = \neg\mathbf{e}$.
- Take the variety $\mathbb{NM}^-$ of NM algebras without a fixed point, axiomatized as an extensions of $\mathbb{NM}$ with the equation

$$\neg((\neg(x \odot x) \odot \neg(x \odot x)) = (\neg(\neg x \odot \neg x)) \odot (\neg(\neg x \odot \neg x)).$$

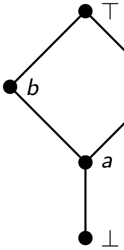
Gödel and NM^{+}

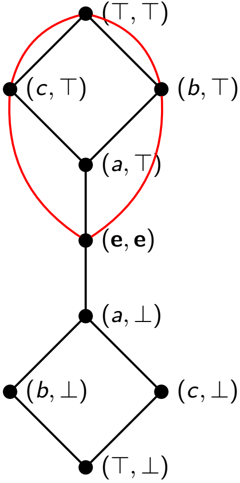
The connected rotation of a Gödel algebra is an NM^{+} algebra.



Gödel and NM^+

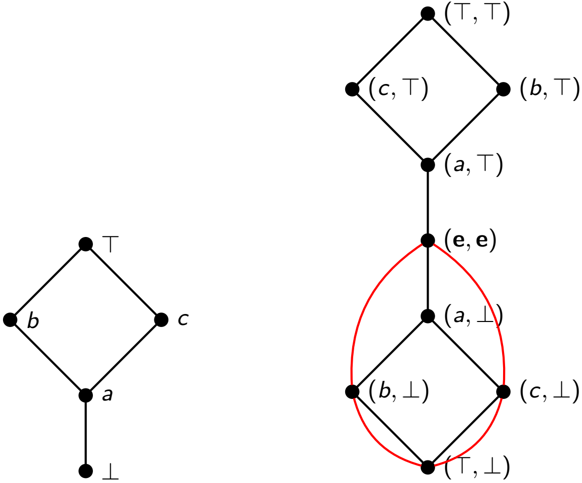
The connected rotation of a Gödel algebra is an NM^+ algebra.



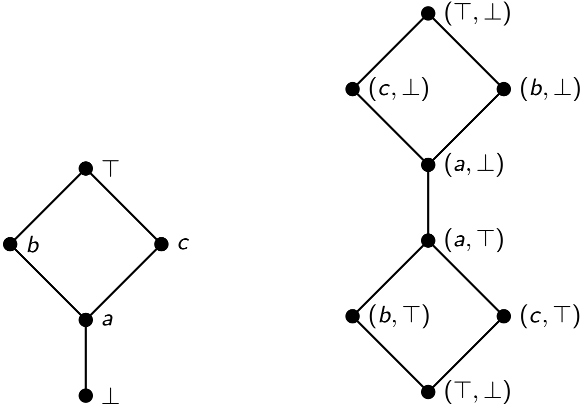


Gödel and NM^+

The connected rotation of a Gödel algebra is an NM^+ algebra.

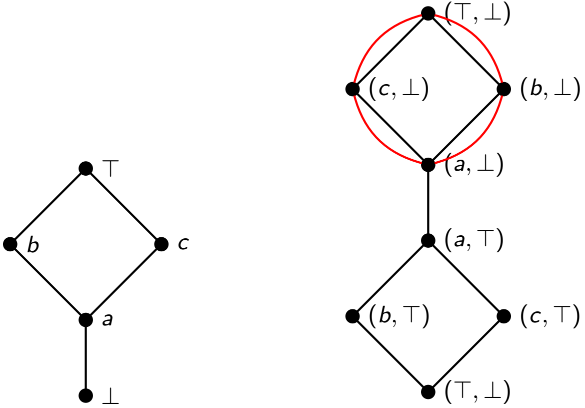


The disconnected rotation of a Gödel algebra is an NM$^-$ algebra.

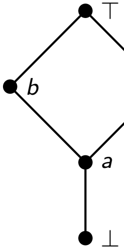


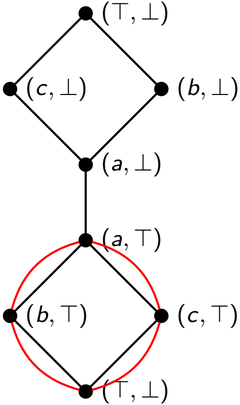
Gödel and NM$^-$

The disconnected rotation of a Gödel algebra is an NM$^-$ algebra.



The disconnected rotation of a Gödel algebra is an NM$^-$ algebra.





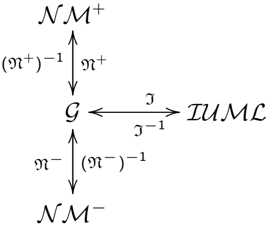
Gödel and NM: the equivalence

An immediate consequence of Busaniche's results:

Theorem

The category of finite Gödel algebras with homomorphisms, is equivalent both to the category of finite NM^+ algebras with homomorphisms and to the category of finite NM^- algebras with homomorphisms.

The Equivalences



Current and Future Works

- Generalize the above equivalences.
- M. Busaniche studied the general case through Weak Boolean Products.
- Study free IUML-algebras.
- Study amalgamation properties.
- Study the concept of state.

THANKS!