Rings and Gödel Algebras

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- We want to study those rings whose semiring of ideals is a Gödel algebra.
- This falls under the category of coordinatization problems.
- [Von Neumann] Any complemented modular lattice L with a finite homogeneous spanning sequence with at least four elements is *coordinatizable*, that is, there exists a von Neumann regular ring R whose lattice of principal right ideals is isomorphic to L.
- [Bergman] Any algebraic distributive lattice with at most $\aleph_0$ compact elements is isomorphic to the lattice of ideals of a Von Neumann regular ring.
- [Wehrung] Every algebraic distributive lattice with at most $\aleph_1$ compact elements is isomorphic to the ideal lattice of a von Neumann regular ring. $\aleph_1$ cannot be replaced by $\aleph_2$.
- [Belluce, Di Nola] A commutative ring R is a Łukasiewicz ring iff it is a direct sum of commutative Artinian chain rings with unit.

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Gödel Rings (I)

- Let $R = \langle R, +, -, \cdot, 0 \rangle$ be a ring, not necessarily commutative nor necessarily with identity.
- We assume that R satisfies the condition
  ($\star$) for all $x \in R$ there are $s, t \in R$ such that $sx = x = xt$.
- Let $Sem(R)$ denote the semiring of (two-sided) ideals of R under ideal sum and ideal product.
- Let $Id(R)$ denote the lattice of ideals of R.
- $Sem(R)$ and $Id(R)$ share the same universe, but they do not coincide.

Gödel Rings (II)

- We want $Sem(R)$ to be a lattice w.r.t. the ideal sum and ideal product.
- Thus, we want rings R to satisfy $IJ = I \cap J$ for all ideals $I, J \in Sem(R)$.
- We want $Id(R)$ and $Sem(R)$ to coincide as sets and as monoidal structures.
- Assume that is the case. $Id(R)$ is an algebraic lattice, and so we can define the residual operator $I \to J = \bigvee\{L \mid IL \subseteq J\}$.
- $Id(R)$ can be naturally seen as a Heyting algebra.

Gödel Rings (III)

- $I \to J$ can be given a standard ideal theoretic interpretation, i.e.

$$(J : I) = \{x \mid Ix \subseteq J\} = \{x \mid xI \subseteq J\}.$$

- We focus on rings where $Id(R)$ satisfies $(J : I) + (I : J) = R$ for all ideals $I, J \subseteq R$.
- Any such a ring will be called a *Gödel ring*.

Examples

- Von Neumann regular (VNR) rings, i.e. rings R such that for all $x \in R$ there is a $y \in R$ such that $x = xyx$.

- For any VNR ring R, $Id(R)$ is a Heyting algebra.

- For a commutative ring R, $Id(R)$ is a Heyting algebra iff R is a VNR ring.

- If R is the full ring of linear transformations on a vector space over a division ring, then R is VNR, and its set of ideals forms a chain [Kaplanski]. Then $Id(R)$ is indeed a Gödel algebra.

- Let R be a ring with unit that has no zero divisors. If R is not a division ring, then R is not VNR.

- Let F be a field of characteristic 0, and let $R = F[x, y]$ be the ring of polynomials $p(x, y) = \Sigma_{ij}\alpha_{ij}x^i y^j$ where x, y commute with the elements of F but $xy - yx = 1$. This ring belongs to the class of Weyl algebras [Herstein].

- Notice that $I \to 0 = I^*$, where I^* is the annihilator of I, i.e. $I^* = \{x \in R \mid yx = 0, \forall y \in I\}$.
- Given a Gödel Ring R, R/I is a Gödel ring.
- A direct sum of Gödel rings is a Gödel ring.
- Every Gödel ring in semiprime (i.e. the zero-ideal is the intersection of prime ideals).
- Every Gödel ring is prime (the monoid of ideals has no zero-divisors) iff it is finitely subdirectly irreducible.

Representation Theorem

Theorem

If R is a Gödel ring, then

1. *R is a subdirect product of subdirectly irreducible prime Gödel rings R_i;*
2. *$Id(R) \hookrightarrow \prod_i Id(R_i)$;*
3. *each $Id(R_i)$ is an algebraic Gödel algebra with a unique atom.*

Proof (Sketch)

- $R \hookrightarrow \prod R/\theta_i$, where each θ_i is a completely meet irreducible congruence relation and $\bigcap\{\theta_i\} = \mathbf{0}_R$.

- $\mathbf{Con}(R/\theta_i)$ has a monolith ν: so $\mathbf{Con}(R/\theta_i)$ has a unique non-zero minimal congruence contained in every other element of $\mathbf{Con}(R/\theta_i)$ and then R/θ_i is prime since, obviously, $\mathbf{Con}(R/\theta_i)$ has no zero-divisors.

- Each θ_i is a strictly meet irreducible element in the lattice of congruences $\mathbf{Con}(R)$ of R.

- Let $[\theta_i)$ be the principal filter in $\mathbf{Con}(R)$ generated by θ_i.

- Since $\mathbf{Con}(R)$ is a distributive lattice, by Ore, the map $\phi_{\theta_i} : x \mapsto \theta_i \vee x$ is a frame homorphism from $\mathbf{Con}(R)$ onto $[\theta_i)$, and the binary relation

$$x \equiv_{\theta_i} y \text{ iff } \theta_i \vee x = \theta_i \vee y$$

is a congruence relation.

Proof (Sketch)

- Easily $\bigcap\{\equiv_{\theta_i}\} = \mathbf{0}_{\mathbf{Con}(R)}$, and so $\mathbf{Con}(R)$ is a subdirect product of the algebraic lattices $[\theta_i)$.
- $[\theta_i)$ is isomorphic to $\mathbf{Con}(R/\theta_i)$. Consequently $\mathbf{Con}(R)$ is a subdirect product of algebraic lattices $\mathbf{Con}(R/\theta_i)$, each with a unique atom.
- It is easy to see that each $\mathbf{Con}(R/\theta_i)$ is a Gödel algebra.
- $Id(R)$ is a subdirect product of algebraic Gödel algebras $Id(R/\theta_i)$ each with a unique atom.

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Spectra of prime ideals (I)

- We study the connection between prime ideals of a Gödel ring and prime complete ideals of its related Gödel algebra.
- Let W be a prime ideal of R, and let $\mathcal{W} = \langle W \rangle$ be the ideal in $Id(R)$ generated by W. $\mathcal{W}$ is prime and complete.
- Conversely, suppose that $\mathcal{W}$ is a complete prime ideal of $Id(R)$ and let $W = \{x \in R \mid RxR \in \mathcal{W}\}$. It is easy to see that W is a prime ideal of R and that $\mathcal{W} = \langle W \rangle$.

Spectra of prime ideals (II)

- Then, consider the mapping

$$\mathcal{P}\colon Spec(R) \to Spec_\sigma(Id(R)) \text{ given by } \mathcal{P}(W) = \langle W \rangle$$

and consider the mapping

$$\mathcal{S}\colon Spec_\sigma(Id(R)) \to Spec(R) \text{ given by } \mathcal{S}(\mathcal{W}) = \{x \in R \mid RxR \in \mathcal{W}\},$$

- These maps are inverse to each other (i.e. $\mathcal{PS}(\mathcal{W}) = \mathcal{W}$ and $\mathcal{SP}(W) = W$) and they also preserve $\subseteq$.

- Both $\mathcal{P}$ and $\mathcal{S}$ are continuous mappings.

Theorem

$Spec(R)$ is homeomorphic to $Spec_\sigma(Id(R))$.

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Ring Representable Gödel Algebras (I)

- For which Gödel algebras A there exists a Gödel ring R such that $Id(R) \cong A$?
- Any algebra of this kind will be called *ring-representable*.
- $Id(R)$ must be an algebraic Gödel algebra, that is, complete and compactly generated.

Ring Representable Gödel Algebras (II)

Theorem

For every finite linearly ordered Gödel algebra G there is a Gödel ring R such that $Id(R) \cong G$.

- Given a finite linearly ordered Gödel algebra G, R coincides with be the full ring of linear transformations on a vector space V over a division ring D of dimension α, where α is the cardinal $\aleph_n$, with $n = |G|$.

- [Wehrung00] If H is an algebraic Heyting lattice such that the set of compact elements has cardinality less than $\aleph_1$, then there is a VNR ring R such that $Id(R) \cong H$.

- The class of ring representable Gödel algebras generates the whole variety of Gödel algebras.

Future Work

- L.P. Belluce, A. Di Nola, E. Marchioni, Rings and Gödel algebras. *Algebra Universalis*, to appear.
- Complete characterization of Gödel Rings.
- Sheaf representation.
- Classes of Gödel Rings.

THANKS!