A Theory of Modal Natural Dualities with Applications to Many-Valued Modal Logics

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Outline

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The theory of natural dualities (TND)

TND discusses:

- When a duality holds for $\mathbb{ISP}(M)$ for a finite algebra M.
  - $\mathbb{ISP}(M) \risingdotseq$ algebras of "M-valued logic".
- Reference: "Natural dualities for the working algebraist" (Davey and Clark, CUP).

TND encompasses:

- Stone duality for the class of Boolean algebras, which is $\mathbb{ISP}(\mathbf{2})$ where $\mathbf{2}$ is the two-element Boolean algebra.
- Priestley duality for the class of distributive lattices, which is $\mathbb{ISP}(\mathbf{2})$ where $\mathbf{2}$ is the two-element distributive lattice.
- Cignoli duality for the class of MV_n algebras, which is $\mathbb{ISP}(\mathbf{n})$ where $\mathbf{n}$ is $\{0, 1/(n-1), ..., 1\}$ (as an MV-algebra).

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TND does not encompass JT or KKV dualities

TND does not encompass:

- Jónsson-Tarski duality for the class of modal algebras, which is not $\mathbb{ISP}(M)$ for any finite algebra M.
  - Teheux duality for modal MV_n algs. generalize this.
- Kupke-Kurz-Venema coalgebraic duality for modal algs..

Some details of the above dualities:

- JT duality: The cat. of modal algs. is dual. equiv. to that of relational Bool. spaces (or descriptive general frames).
- KKV duality: The cat. of modal algs. is dual. equiv. to that of $\mathbb{V}$-coalgs. for Vietoris functor $\mathbb{V}$ on the cat. of Bool. sp..

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We extend TND by modalizing $\mathbb{ISP}$

We extend TND so that it encompasses JT and KKV dualities.

- This is done by introducing the notion of $\mathbb{ISP}_{\mathbb{M}}$.
  - $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ = (all) modal algs.. $\mathbb{ISP}_{\mathbb{M}}(\mathbf{n})$ = modal MV_n alg..
- Thus we develop duality theory for $\mathbb{ISP}_{\mathbb{M}}(L)$ to understand and generalize JT and KKV dualities from the viewpoint of TND, with appl. to MV modal logics.

The main result (details are given later):

- $\mathbf{ISP_M}(L) \simeq \mathbf{RBS}_L^{\mathrm{op}} \simeq \mathbf{Coalg}(\mathrm{V}_L)^{\mathrm{op}}$ where L is a quasi-primal algebra with a bounded lattice reduct.
  - By letting $L = \mathbf{2}$, we can recover JT and KKV dualities.
- Dualities for MV modal logics follow from these results (Teheux duality and some new dualities).

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Comparison with related work

Related work: [Priestley;J.Austral.Math.Soc.(A)63;1997], [Davey,Talukder;preprint;2009].

- They consider duality theory for $\mathbb{ISP}(M)$ where M is a finite alg. with unary operators (seen as modalities).
- But, $\mathbb{ISP}(M) \neq$ the class of (all) modal algebras.
- So they cannot encompass JT nor KKV dualities.
- Our duality theory for $\mathbb{ISP}_{\mathbb{M}}(L)$ does encompass JT, Teheux, and KKV dualities.

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The notion of modal power

$L :=$ a finite algebra with a lattice reduct.
$L^S :=$ the set of all functions from S to L.

Definition (modal power w.r.t. Kripke frame)

For a Kripke frame (S, R), the modal power of L w.r.t. (S, R) is $L^S \in \mathbb{ISP}(L)$ equipped with an operation $\Box_R$ on L^S defined by

$$(\Box_R f)(w) = \bigwedge \{f(w') \,;\, wRw'\}$$

where $f \in L^S$ and $w \in S$.

Without a lattice reduct: replace $\bigwedge$ with a unary operation on L.

- The notion of intuitionistic power can be similarly defined, which is useful to incorporate Esakia duality for Heyt. algs. into TND.

$\mathbb{ISP}$ Modalized

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A modal power of L is defined as
the modal power of L w.r.t. (S, R) for a Kripke frame (S, R).

Definition ($\mathbb{ISP}_{\mathbb{M}}$)

$\mathbb{ISP}_{\mathbb{M}}(L)$ denotes the class of all isomorphic copies of subalgebras of modal powers of L.

- $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ = the class of modal algebras.
- $\mathbb{ISP}_{\mathbb{M}}(\mathbf{n})$ = the class of modal MV_n algebras, which were introduced by Hansoul and Teheux in 2006.

$\mathbb{ISP}_{\mathbb{M}}(L) \doteqdot$ algebras of modal L-valued logic.

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The notion of Kripke condition

For $(A,\Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$, define a relation $R_\Box$ on $\mathrm{Hom}_{\mathbf{ISP}(L)}(A,L)$:

Definition ($R_\Box$)

For $v, u \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A,L)$, $vR_\Box u$ iff
$\forall a \in L\ \forall x \in A\ (v(\Box x) \geq a$ implies $u(x) \geq a)$.

Definition (Kripke condition)

$\mathbb{ISP}_{\mathbb{M}}(L)$ satisfies the Kripke condition iff
for any $(A,\Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$ and $v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A,L)$,

$$v(\Box x) = \bigwedge \{u(x)\ ;\ vR_\Box u\} \text{ for } x \in A.$$

- $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ satisfies the Kripke condition.
- $\mathbb{ISP}_{\mathbb{M}}(\mathbf{n})$ ($=$ modal MV_n algs.) satisfies the Kripke cond..

Kripke condition $\fallingdotseq$ Kripke completeness.

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Ternary Discriminator

$L :=$ a finite algebra. An important notion in universal algebra:

Definition (Pixely, Werner, 1970)

The ternary discriminator $d : L^3 \to L$ on L is defined by:

$$d(x, y, z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{if } x = y \end{cases}$$

where $x, y, z \in L$.

We assume in the following part of this talk:

- L has a bounded lattice reduct.
- L is quasi-primal, i.e., d is a term function of L.

Then L is semi-primal (we omit its definition).

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Closedness under $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{H}$

$K :=$ a finite alg. with a bounded lattice reduct.

Proposition

$\mathbb{ISP}_{\mathbb{M}}(K)$ is closed under $\mathbb{P}, \mathbb{S}, \mathbb{I}$.
Hence, $\mathbb{ISP}_{\mathbb{M}}(K)$ has free algebras.

Proposition

If K is quasi-primal, then $\mathbb{ISP}_{\mathbb{M}}(K)$ forms a variety.

- This can be obtained as a corollary of our duality theorem.
- We can also provide a concrete axiomatization of $\mathbb{ISP}_{\mathbb{M}}(K)$ if K is quasi-primal.

Semi-primal duality in TND

$\mathrm{SubAlg}(L)$:= the set of subalgebras of L.
$\mathrm{SubSp}(S)$:= the set of closed subspaces of a Bool. space S.

Definition (Category $\mathbf{BS}_L$)

An object in $\mathbf{BS}_L$ is (S, α) s.t. S is a Bool. sp. and $\alpha : \mathrm{SubAlg}(L) \to \mathrm{SubSp}(S)$ with $S = \alpha(L)$ satisfies:

- $L_1 \subset L_2$ for $L_1, L_2 \in \mathrm{SubAlg}(L)$ implies $\alpha(L_1) \subset \alpha(L_2)$;
- $L_3 = L_1 \cap L_2$ implies $\alpha(L_3) = \alpha(L_1) \cap \alpha(L_2)$.

An arrow $f : (S, \alpha) \to (S', \beta)$ in $\mathbf{BS}_L$ is a conti. map $f : S \to S'$ that satisfies: $\forall M \in \mathrm{SubAlg}(L)\ (x \in \alpha(M) \Rightarrow f(x) \in \beta(M))$.

Theorem (Semi-primal duality in TND)

$\mathbf{ISP}(L) \simeq \mathbf{BS}_L^{\mathrm{op}}$. *Cignoli duality for* MV_n*-alg. is the case* $L = \mathbf{n}$.

Outline

Introduction

New Notion: $\mathbb{ISP}_{\mathbb{M}}(L)$

Main Result: Coalgebraic Duality for $\mathbb{ISP}_{\mathbb{M}}(L)$

Category $\mathbf{RBS}_L$

For a Kripke frame (S, R) and $X \subset S$,
$R^{-1}[X] := \{w \in S\,;\ \exists w' \in X\ wRw'\}$. $R[w] := \{w' \in S\,;\ wRw'\}$.

Definition (Category $\mathbf{RBS}_L$)

An object in $\mathbf{RBS}_L$ is (S, α, R) such that (S, α) is in $\mathbf{BS}_L$ and a relation R on S satisfies:

- $R[w]$ is closed in S for any $w \in S$;
- $R^{-1}[X]$ is clopen for any clopen X;
- $\forall M \in \mathrm{SubAlg}(L)$ $(w \in \alpha(M)$ implies $R[w] \subset \alpha(M))$.

An arrow $f : (S_1, \alpha_1, R_1) \to (S_2, \alpha_2, R_2)$ in $\mathbf{RBS}_L$ is an arrow $f : (S_1, \alpha_1) \to (S_2, \alpha_2)$ in $\mathbf{BS}_L$ satisfying

- if wR_1w' then $f(w)R_2f(w')$;
- if $f(w_1)R_2w_2$ then $\exists w' \in S_1$ $(w_1R_1w'$ and $f(w') = w_2)$.

Modal semi-primal duality for $\mathbb{ISP}_{\mathbb{M}}(L)$

For $(A, \Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$,

- $\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$ is equipped with $R_\Box$.

For $(S, \alpha, R) \in \mathbf{RBS}_L$,

- $\mathrm{Hom}_{\mathbf{BS}_L}((S, \alpha), (L, \mathrm{id}))$ is equipped with $\Box_R$.

$\mathbb{ISP}_{\mathbb{M}}(L)$ satisfies the Kripke condition.

- $\fallingdotseq$ completeness of L-valued modal logic.

Theorem (Modal semi-primal duality for $\mathbb{ISP}_{\mathbb{M}}(L)$)

$\mathbf{ISP}_{\mathbf{M}}(L) \simeq \mathbf{RBS}_L^{\mathrm{op}}$.

- JT duality for modal algebras is the case $L = \mathbf{2}$.

$\mathcal{O}_S :=$ the set of opens in S. $\mathcal{C}_S :=$ the set of closed sets in S.

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Coalgebraic duality for $\mathbb{ISP}_{\mathbb{M}}(L)$

Definition (Vietoris space $\mathrm{V}(S)$)

Vietoris space $\mathrm{V}(S)$ of a topo. sp. S is $\mathcal{C}_S$ with the topology generated by $\{B_S(U)\,;\ U \in \mathcal{O}_S\} \cup \{D_S(U)\,;\ U \in \mathcal{O}_S\}$ where $B_S(U) := \{F \in \mathcal{C}_S\,;\ F \subset U\}$, $D_S(U) := \{F \in \mathcal{C}_S\,;\ F \cap U \neq \emptyset\}$.

Definition (L-Vietoris functor $\mathrm{V}_L : \mathbf{BS}_L \to \mathbf{BS}_L$)

Object: $\mathrm{V}_L(S, \alpha) := (\mathrm{V}(S), \mathrm{V} \circ \alpha)$.
Arrow: $\mathrm{V}_L(f)$ is defined by $\mathrm{V}_L(f)(F) = f(F)$ for $F \in \mathrm{V}(S)$.

Theorem (Coalg. duality for $\mathbb{ISP}_{\mathbb{M}}(L)$)

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MV modal logics

Form$_\Box$ denotes the set of formulas constructed by $\Box$, the connectives of Łukasiewicz logic, and variables.

Definition (Hansoul and Teheux 2006)

Let (W, R) be a Kripke frame (i.e., R is a relation on a set W). Then $e : W \times \mathbf{Form}_\Box \to \mathbf{n}$ is a Kripke **n**-valuation on (W, R) iff for each $w \in W$ and $\varphi, \psi \in \mathbf{Form}_\Box$,

- $e(w, \Box\varphi) = \bigwedge\{e(w', \varphi)\ ;\ wRw'\}$;
- $e(w, \varphi@\psi) = e(w, \varphi)@e(w, \psi)$ for $@ = \wedge, \vee, *, \wp, \to$;

Then, (W, R, e) is called an **n**-valued Kripke model.

Define $\mathrm{MŁ}_n$ as the set of formulas $\varphi \in \mathbf{Form}_\Box$ such that $e(w, \varphi) = 1$ for any **n**-val. Kripke model (W, R, e) and $w \in W$.

- Modal MV_n-algebras := algebras of $\mathrm{MŁ}_n$.

Applications to MV modal logics

Since **n** is quasi-primal and $\mathbb{ISP}_{\mathbb{M}}(\mathbf{n}) =$ modal MV_n-algs., Teheux duality below is a corollary of the first result.

Theorem (Teheux, 2007)

The cat. of modal MV_n-algs. is dually equiv. to $\mathbf{RBS_n}$.

We can obtain a compactness thm. for $\mathrm{MŁ}_n$ from this duality.

- The second result gives us a new duality:

Theorem (Coalg. duality for modal MV_n-algs.)

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By this we obtain: $\exists F : \mathbf{MV}_n \to \mathbf{MV}_n$ $Alg(F) \simeq$ modal MV_n-algs..

- We can obtain similar results for a Fitting's MV modal logic. Some MV modal logics can be treated in a uniform way.

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Conclusions

Results of this work:

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Significance of this work:

- We extended TND to encompass JT, Teheux, KKV duals..
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That's All.

Thank you for your attention!!