Proof Theory for Fuzzy Logics

A Tutorial: Part Two

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Last Time

- We have seen that certain logics (classes of algebras) can be presented algorithmically in the framework of *Gentzen systems*.
- In particular, many fuzzy logics (classes of semilinear BPCRLs) can be presented in the framework of *hypersequent calculi*.
- Let us remind ourselves (quickly!) how this works...

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The Algebraic Framework

Definition

A *bounded pointed commutative residuated lattice (BPCRL)* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, \mathrm{e}, \mathrm{f}, \bot, \top \rangle$ such that:

1. $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice;
2. $\langle A, \cdot, \mathrm{e} \rangle$ is a commutative monoid;
3. $z \leq x \to y$ iff $x \cdot z \leq y$ for all $x, y, z \in A$.

A *semilinear BPCRL* satisfies additionally:

$$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z) \qquad \text{(distributivity)}$$

$$\mathrm{e} \leq (x \to y) \vee (y \to x) \qquad \text{(prelinearity).}$$

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A Proof System for Lattices

Axioms

$$\frac{}{\varphi \Rightarrow \varphi}\ (\text{ID})$$

Cut rule

$$\frac{\psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\psi \Rightarrow \varphi}\ (\text{CUT})$$

Left logical rules

$$\frac{\varphi_i \Rightarrow \psi}{\varphi_1 \wedge \varphi_2 \Rightarrow \psi}\ (\wedge\Rightarrow)_i \quad i = 1, 2$$

$$\frac{\varphi_1 \Rightarrow \psi \quad \varphi_2 \Rightarrow \psi}{\varphi_1 \vee \varphi_2 \Rightarrow \psi}\ (\vee\Rightarrow)$$

Right logical rules

$$\frac{\psi \Rightarrow \varphi_1 \quad \psi \Rightarrow \varphi_2}{\psi \Rightarrow \varphi_1 \wedge \varphi_2}\ (\Rightarrow\wedge)$$

$$\frac{\psi \Rightarrow \varphi_i}{\psi \Rightarrow \varphi_1 \vee \varphi_2}\ (\Rightarrow\vee)_i \quad (\text{i} = 1,2)$$

A Sequent Calculus for Lattices

Axioms

$$\frac{}{\varphi \Rightarrow \varphi} \text{ (ID)}$$

Cut rule

$$\frac{\Gamma_1 \Rightarrow \chi \quad \Gamma_2, \chi \Rightarrow \Delta}{\Gamma_1, \Gamma_2 \Rightarrow \Delta} \text{ (CUT)}$$

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$$\frac{\Gamma, \varphi_i \Rightarrow \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta} \ (\wedge\Rightarrow)_i \quad i = 1, 2$$

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A Sequent Calculus for BPCRLs

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$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \cdot \psi \Rightarrow \Delta} \ (\cdot \Rightarrow)$$

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$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \mathrm{e} \Rightarrow \Delta} \ (\mathrm{e} \Rightarrow) \qquad \frac{}{\mathrm{f} \Rightarrow} \ (\mathrm{f} \Rightarrow)$$

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$$\frac{\Gamma_1 \Rightarrow \varphi \quad \Gamma_2 \Rightarrow \psi}{\Gamma_1, \Gamma_2 \Rightarrow \varphi \cdot \psi} \ (\Rightarrow \cdot)$$

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightarrow \psi} \ (\Rightarrow\rightarrow)$$

$$\frac{}{\Rightarrow \mathrm{e}} \ (\Rightarrow \mathrm{e}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathrm{f}} \ (\Rightarrow \mathrm{f})$$

$$\frac{}{\Gamma \Rightarrow \top} \ (\Rightarrow\top)$$

A Hypersequent Calculus for BPCRLs

Axioms

$$\frac{}{\mathcal{G} \mid \varphi \Rightarrow \varphi}\ (\text{ID})$$

Cut rule

$$\frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \chi \quad \mathcal{G} \mid \Gamma_2, \chi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta}\ (\text{CUT})$$

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Hypersequent Calculi

We obtain a hypersequent calculus GUL for **semilinear BPCRLs** by adding *external weakening* and *external contraction* rules:

$$\frac{\mathcal{G}}{\mathcal{G} \mid \mathcal{H}}\ (\text{EW}) \qquad \frac{\mathcal{G} \mid \mathcal{H} \mid \mathcal{H}}{\mathcal{G} \mid \mathcal{H}}\ (\text{EC})$$

and the *communication* rule:

$$\frac{\mathcal{G} \mid \Gamma_1, \Pi_1 \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2, \Pi_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1 \mid \Pi_1, \Pi_2 \Rightarrow \Delta_2}\ (\text{COM})$$

Further calculi are obtained by adding *structural rules* such as:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Delta}\ (\text{WL}) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (\text{WR}) \qquad \frac{\mathcal{G} \mid \Gamma, \Pi, \Pi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Delta}\ (\text{CL})$$

E.g., GMTL is GUL + (WL) + (WR) and GG is GMTL + (CL).

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Today

But are hypersequent calculi *useful* for anything?

- We use density elimination to tackle the *standard completeness* problem for fuzzy logics.
- We extend the calculi to *first-order fuzzy logics* and prove a Herbrand theorem.
- We use a hypersequent calculus to prove *decidability* and *PSPACE-completeness* for a Gödel modal logic.

Also, what about the *really important* (fundamental) fuzzy logics?

- We present "non-standard" hypersequent calculi for *Łukasiewicz logic*, *cancellative hoop logic*, and *product logic*.
- We confess our ignorance for *basic logic* and pose some other open problems.

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Standard Completeness

Each variety of semilinear BPCRLs is generated by its *linearly ordered* members (chains). But is such a variety also generated by

(1) its *dense* linearly ordered members?

(2) its *standard* members of the form

$$\langle [0, 1], \min, \max, *, \to_*, e_*, f, 0, 1 \rangle \ ?$$

(3) one special standard algebra?

We can use our hypersequent calculi to tackle (1).

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The Basic Idea

Extend GUL with a "density rule" to obtain GUL^{D} and show:

$\varphi \leq \psi$ in all semilinear BPCRLs

iff $\varphi \Rightarrow \psi$ is derivable in GUL

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The Density Rule

Let GUL^D be GUL extended with:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow x \mid \Pi, x \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Delta} \text{ (DENSITY)}$$

where x does not occur in the conclusion.

A formula version of this rule first appeared in

G. Takeuti and T. Titani. Intuitionistic fuzzy logic and intuitionistic fuzzy set theory. *Journal of Symbolic Logic*, 49(3):851–866, 1984.

Theorem

$\vdash_{\text{GUL}^D} \varphi \Rightarrow \psi$ *iff* $\varphi \leq \psi$ *in all dense semilinear BPCRLs.*

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What Can Go Wrong?

A calculus GCL for classical logic is obtained by extending GMTL with

$$\frac{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow} \ (\text{SPLIT})$$

But then for any formula φ, we have a derivation in GCL^{D}:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x \Rightarrow x}\ (\text{ID})}{\Rightarrow x \mid x \Rightarrow}\ (\text{SPLIT})}{\Rightarrow}\ (\text{DENSITY})}{\Rightarrow \varphi}\ (\text{WR})$$

I.e., GCL^{D} is *trivial* – as it should be.

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What Can Go Wrong?

A calculus GCL for classical logic is obtained by extending GMTL with

$$\frac{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow} \ (\textsc{split})$$

But then for any formula φ, we have a derivation in GCL^{D}:

$$\cfrac{\cfrac{\cfrac{\overline{x \Rightarrow x}\ (\textsc{id})}{\Rightarrow x \mid x \Rightarrow}\ (\textsc{split})}{\Rightarrow}\ (\textsc{density})}{\Rightarrow \varphi}\ (\textsc{wr})$$

I.e., GCL^{D} is *trivial* – as it should be.

Density Elimination (1)

Suppose that we have a (cut and density free) derivation ending in:

$$\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow x \mid \Pi, x \Rightarrow \Delta}}{\Gamma, \Pi \Rightarrow \Delta} \ (\text{DENSITY})$$

Replace x on the *left* by Γ and on the *right* by Π (left) and Δ (right):

$$\dfrac{\dfrac{\vdots}{\Gamma, \Pi \Rightarrow \Delta \mid \Gamma, \Pi \Rightarrow \Delta}}{\Gamma, \Pi \Rightarrow \Delta} \ (\text{EC})$$

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Density Elimination (2)

However, we could have a derivation:

$$\dfrac{\dfrac{\overline{x \Rightarrow x}\ (\text{ID}) \qquad \dfrac{\vdots}{\Gamma, \Pi \Rightarrow \Delta}}{\Gamma \Rightarrow x \mid \Pi, x \Rightarrow \Delta}\ (\text{COM})}{\Gamma, \Pi \Rightarrow \Delta}\ (\text{DENSITY})$$

Replacing xs as before, we get the unhelpful:

$$\dfrac{\dfrac{\Gamma, \Pi \Rightarrow \Delta \qquad \dfrac{\vdots}{\Gamma, \Pi \Rightarrow \Delta}}{\Gamma, \Pi \Rightarrow \Delta \mid \Gamma, \Pi \Rightarrow \Delta}\ (\text{COM})}{\Gamma, \Pi \Rightarrow \Delta}\ (\text{EC})$$

Clearly here we can replace the application of (COM) with (EW). More generally, we can use (CUT) and cut elimination to repair derivations. . .

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A Recap

$\varphi \leq \psi$ in all semilinear BPCRLs

iff $\varphi \Rightarrow \psi$ is derivable in GUL

iff $\varphi \Rightarrow \psi$ is derivable in GUL^{D}

iff $\varphi \leq \psi$ in all dense linearly ordered semilinear BPCRLs.

A. Ciabattoni and G. Metcalfe. Density elimination.
Theoretical Computer Science, 403:328–346, 2008.

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First-Order Fuzzy Logics

- Extending hypersequent calculi to first-order logics is relatively straightforward... we just add rules for the quantifiers $\forall$ and $\exists$.
- We consider terms s, t and formulas φ, ψ of usual first-order languages, but for convenience, we make a syntactic distinction between *bound* variables x, y and *free* variables a, b.

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First-Order Hypersequent Calculi

GUL∀ consists of the hypersequent calculus GUL extended to first-order formulas, plus the quantifier rules:

$$\frac{\mathcal{G} \mid \Gamma, \varphi(t) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\forall x)\varphi(x) \Rightarrow \Delta} \ (\forall\Rightarrow) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(a)}{\mathcal{G} \mid \Gamma \Rightarrow (\forall x)\varphi(x)} \ (\Rightarrow\forall)$$

$$\frac{\mathcal{G} \mid \Gamma, \varphi(a) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\exists x)\varphi(x) \Rightarrow \Delta} \ (\exists\Rightarrow) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(t)}{\mathcal{G} \mid \Gamma \Rightarrow (\exists x)\varphi(x)} \ (\Rightarrow\exists)$$

where a does not occur in the conclusions of $(\Rightarrow\forall)$ or $(\exists\Rightarrow)$.

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where *a* does not occur in the conclusions of $(\Rightarrow\forall)$ or $(\exists\Rightarrow)$.

An Example Derivation

$$\dfrac{\dfrac{}{\psi(a) \Rightarrow \psi(a)}\ (\text{ID}) \qquad \dfrac{}{\varphi \Rightarrow \varphi}\ (\text{ID})}{\psi(a) \Rightarrow \varphi \mid \varphi \Rightarrow \psi(a)}\ (\text{COM}) \qquad \dfrac{}{\psi(a) \Rightarrow \varphi \mid \psi(a) \Rightarrow \psi(a)}\ (\text{ID})$$

$$\dfrac{\psi(a) \Rightarrow \varphi \mid \varphi \Rightarrow \psi(a) \qquad \psi(a) \Rightarrow \varphi \mid \psi(a) \Rightarrow \psi(a)}{\psi(a) \Rightarrow \varphi \mid \varphi \vee \psi(a) \Rightarrow \psi(a)}\ (\vee\Rightarrow)$$

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$$\dfrac{\varphi \vee \psi(a) \Rightarrow \varphi \mid \varphi \vee \psi(a) \Rightarrow \psi(a)}{\varphi \vee \psi(a) \Rightarrow \varphi \mid (\forall x)(\varphi \vee \psi) \Rightarrow \psi(a)}\ (\forall\Rightarrow)$$

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$$\dfrac{(\forall x)(\varphi \vee \psi) \Rightarrow \varphi \mid (\forall x)(\varphi \vee \psi) \Rightarrow \varphi \vee (\forall x)\psi}{(\forall x)(\varphi \vee \psi) \Rightarrow \varphi \vee (\forall x)\psi \mid (\forall x)(\varphi \vee \psi) \Rightarrow \varphi \vee (\forall x)\psi}\ (\Rightarrow\vee)_1$$

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$$\dfrac{(\forall x)(\varphi \vee \psi) \Rightarrow \varphi \vee (\forall x)\psi}{\Rightarrow (\forall x)(\varphi \vee \psi) \rightarrow (\varphi \vee (\forall x)\psi)}\ (\Rightarrow\rightarrow)$$

Note that, by definition, x is not free in φ.

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi \mid \varphi \lor \psi(a) \Rightarrow \psi(a)}\,(\text{ID}) \quad \dfrac{\dfrac{\dfrac{}{\psi(a) \Rightarrow \psi(a)}\,(\text{ID}) \quad \dfrac{}{\varphi \Rightarrow \varphi}\,(\text{ID})}{\psi(a) \Rightarrow \varphi \mid \varphi \Rightarrow \psi(a)}\,(\text{COM}) \quad \dfrac{}{\psi(a) \Rightarrow \varphi \mid \psi(a) \Rightarrow \psi(a)}\,(\text{ID})}{\psi(a) \Rightarrow \varphi \mid \varphi \lor \psi(a) \Rightarrow \psi(a)}\,(\lor\Rightarrow)}{\varphi \lor \psi(a) \Rightarrow \varphi \mid \varphi \lor \psi(a) \Rightarrow \psi(a)}\,(\lor\Rightarrow)}{\varphi \lor \psi(a) \Rightarrow \varphi \mid (\forall x)(\varphi \lor \psi) \Rightarrow \psi(a)}\,(\forall\Rightarrow)}{(\forall x)(\varphi \lor \psi) \Rightarrow \varphi \mid (\forall x)(\varphi \lor \psi) \Rightarrow \psi(a)}\,(\forall\Rightarrow)}{(\forall x)(\varphi \lor \psi) \Rightarrow \varphi \mid (\forall x)(\varphi \lor \psi) \Rightarrow (\forall x)\psi}\,(\Rightarrow\forall)}{(\forall x)(\varphi \lor \psi) \Rightarrow \varphi \mid (\forall x)(\varphi \lor \psi) \Rightarrow \varphi \lor (\forall x)\psi}\,(\Rightarrow\lor)_2}{(\forall x)(\varphi \lor \psi) \Rightarrow \varphi \lor (\forall x)\psi \mid (\forall x)(\varphi \lor \psi) \Rightarrow \varphi \lor (\forall x)\psi}\,(\Rightarrow\lor)_1}{(\forall x)(\varphi \lor \psi) \Rightarrow \varphi \lor (\forall x)\psi}\,(\text{EC})}{\Rightarrow (\forall x)(\varphi \lor \psi) \to (\varphi \lor (\forall x)\psi)}\,(\Rightarrow\to)
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An Example Derivation

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$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(ID)}
\UnaryInfC{$\varphi \Rightarrow \varphi \mid \varphi \vee \psi(a) \Rightarrow \psi(a)$}
\AxiomC{}
\RightLabel{(ID)}
\UnaryInfC{$\psi(a) \Rightarrow \psi(a)$}
\AxiomC{}
\RightLabel{(ID)}
\UnaryInfC{$\varphi \Rightarrow \varphi$}
\RightLabel{(COM)}
\BinaryInfC{$\psi(a) \Rightarrow \varphi \mid \varphi \Rightarrow \psi(a)$}
\AxiomC{}
\RightLabel{(ID)}
\UnaryInfC{$\psi(a) \Rightarrow \varphi \mid \psi(a) \Rightarrow \psi(a)$}
\RightLabel{$(\vee\Rightarrow)$}
\BinaryInfC{$\psi(a) \Rightarrow \varphi \mid \varphi \vee \psi(a) \Rightarrow \psi(a)$}
\RightLabel{$(\vee\Rightarrow)$}
\BinaryInfC{$\varphi \vee \psi(a) \Rightarrow \varphi \mid \varphi \vee \psi(a) \Rightarrow \psi(a)$}
\RightLabel{$(\forall\Rightarrow)$}
\UnaryInfC{$\varphi \vee \psi(a) \Rightarrow \varphi \mid (\forall x)(\varphi \vee \psi) \Rightarrow \psi(a)$}
\RightLabel{$(\forall\Rightarrow)$}
\UnaryInfC{$(\forall x)(\varphi \vee \psi) \Rightarrow \varphi \mid (\forall x)(\varphi \vee \psi) \Rightarrow \psi(a)$}
\RightLabel{$(\Rightarrow\forall)$}
\UnaryInfC{$(\forall x)(\varphi \vee \psi) \Rightarrow \varphi \mid (\forall x)(\varphi \vee \psi) \Rightarrow (\forall x)\psi$}
\RightLabel{$(\Rightarrow\vee)_2$}
\UnaryInfC{$(\forall x)(\varphi \vee \psi) \Rightarrow \varphi \mid (\forall x)(\varphi \vee \psi) \Rightarrow \varphi \vee (\forall x)\psi$}
\RightLabel{$(\Rightarrow\vee)_1$}
\UnaryInfC{$(\forall x)(\varphi \vee \psi) \Rightarrow \varphi \vee (\forall x)\psi \mid (\forall x)(\varphi \vee \psi) \Rightarrow \varphi \vee (\forall x)\psi$}
\RightLabel{(EC)}
\UnaryInfC{$(\forall x)(\varphi \vee \psi) \Rightarrow \varphi \vee (\forall x)\psi$}
\RightLabel{$(\Rightarrow\to)$}
\UnaryInfC{$\Rightarrow (\forall x)(\varphi \vee \psi) \to (\varphi \vee (\forall x)\psi)$}
\end{prooftree}
$$

Note that, by definition, x is not free in φ.

An Example Derivation

$$\dfrac{\dfrac{}{\psi(a) \Rightarrow \psi(a)}\ (\text{ID}) \qquad \dfrac{}{\varphi \Rightarrow \varphi}\ (\text{ID})}{\psi(a) \Rightarrow \varphi \mid \varphi \Rightarrow \psi(a)}\ (\text{COM}) \qquad \dfrac{}{\psi(a) \Rightarrow \varphi \mid \psi(a) \Rightarrow \psi(a)}\ (\text{ID})$$

$$\dfrac{\psi(a) \Rightarrow \varphi \mid \varphi \Rightarrow \psi(a) \qquad \psi(a) \Rightarrow \varphi \mid \psi(a) \Rightarrow \psi(a)}{\psi(a) \Rightarrow \varphi \mid \varphi \vee \psi(a) \Rightarrow \psi(a)}\ (\vee\Rightarrow)$$

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$$\dfrac{(\forall x)(\varphi \vee \psi) \Rightarrow \varphi \vee (\forall x)\psi \mid (\forall x)(\varphi \vee \psi) \Rightarrow \varphi \vee (\forall x)\psi}{(\forall x)(\varphi \vee \psi) \Rightarrow \varphi \vee (\forall x)\psi}\ (\text{EC})$$

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\qquad
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\quad
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Some Remarks

- *Completeness* with respect to corresponding classes of algebras can be proved with a usual Henkin-style construction.
- *Cut-elimination* and *density elimination* proceed as in the propositional case, so we can obtain *standard completeness* results for first-order fuzzy logics.
- We can also use these calculi to prove a *Herbrand theorem* and *Skolemization*.

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Herbrand Universe

For a formula φ, let

- $\mathcal{C}$ be the set of constants occurring in φ (adding one if empty)
- $\mathcal{F}$ be the set of function symbols occurring in φ.

We define

$$\begin{aligned} H_0(\varphi) &= \mathcal{C} \\ H_{n+1}(\varphi) &= H_n(\varphi) \cup \{f(\bar{t}) \mid f \in \mathcal{F} \text{ and } \bar{t} \in H_n(\varphi)\}. \end{aligned}$$

Then the *Herbrand universe* of φ is

$$H(\varphi) = \bigcup_{n=0}^{\infty} H_n(\varphi).$$

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A Herbrand Theorem

A *prenex hypersequent* contains only prenex formulas (i.e., with all quantifiers at the front of the formula).

Theorem (Mid-Hypersequent Theorem)

Let $\mathcal{G}$ be a prenex hypersequent. If $\vdash_{\mathrm{GUL}\forall} \mathcal{G}$, then $d \vdash_{\mathrm{GUL}\forall} \mathcal{G}$ where no propositional inference is below a quantifier inference in d.

Corollary (Herbrand Theorem)

For a quantifier-free formula φ:

$$\vdash_{\mathrm{GUL}\forall} \Rightarrow (\exists \bar{x})\varphi(\bar{x}) \quad \textit{iff} \quad \vdash_{\mathrm{GUL}\forall} \Rightarrow \bigvee_{i=1}^{n} \varphi(\bar{t}_i) \ \textit{for some}\ \bar{t}_1, \ldots, \bar{t}_n \in H(\varphi).$$

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Mid-Hypersequent Theorem Proof Sketch

The Idea. Push (inductively) applications of quantifier rules down and logical rules up in derivations. E.g., to push (COM) above $(\forall\Rightarrow)$:

$$\dfrac{\dfrac{\dfrac{\vdots\, d_1}{\mathcal{G} \mid \Gamma_1, \varphi(t), \Pi_1 \Rightarrow \Delta_1}}{\mathcal{G} \mid \Gamma_1, (\forall x)\varphi(x), \Pi_1 \Rightarrow \Delta_1}\ (\forall\Rightarrow) \qquad \dfrac{\vdots\, d_2}{\mathcal{G} \mid \Gamma_2, \Pi_2 \Rightarrow \Delta_2}}{\mathcal{G} \mid \Gamma_1, (\forall x)\varphi(x), \Gamma_2 \Rightarrow \Delta_1 \mid \Pi_1, \Pi_2 \Rightarrow \Delta_2}\ (\text{COM})$$

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Decidability Again

- Recall that *sequent calculi* can be an invaluable tool for proving decidability / complexity results for logics (classes of algebras).
- *Hypersequent calculi* are not always so useful...
- However, there are cases where the existence of a suitable calculus can be very helpful.

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Modal Fuzzy Logics

Modal fuzzy logics aim to formalize fuzzy modal notions such as "John might be tall", "It will rain tomorrow", "I believe that Mary is happy".

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Let us consider the *Gödel modal logics* GK and GK^F studied in:

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Kripke Models for GK

Formulas *Fm* are based on a standard language for Gödel logic with connectives $\wedge, \vee, \to, \bot$, extended with modalities $\Box$ and $\Diamond$.

Kripke models for GK are triples $\langle W, R, V\rangle$ such that $W \neq \emptyset$, $R \subseteq W^2$, and $V : Fm \times W \to [0, 1]$ satisfies

$$
\begin{aligned}
V(\bot, w) &= 0\\
V(\varphi \wedge \psi, w) &= \min(V(\varphi, w), V(\psi, w))\\
V(\varphi \vee \psi, w) &= \max(V(\varphi, w), V(\psi, w))\\
V(\varphi \to \psi, w) &= V(\varphi, w) \to_{\mathrm{G}} V(\psi, w)\\
V(\Box\varphi, w) &= \inf_{u \in W}(\{1\} \cup \{V(\varphi, u) : Rwu\})\\
V(\Diamond\varphi, w) &= \sup_{u \in W}(\{0\} \cup \{V(\varphi, u) : Rwu\}).
\end{aligned}
$$

where $x \to_{\mathrm{G}} y = y$ if $x > y$; 1 otherwise.

Kripke Models for GK

Formulas *Fm* are based on a standard language for Gödel logic with connectives $\wedge, \vee, \rightarrow, \bot$, extended with modalities $\Box$ and $\Diamond$.

Kripke models for GK are triples $\langle W, R, V \rangle$ such that $W \neq \emptyset$, $R \subseteq W^2$, and $V : Fm \times W \rightarrow [0,1]$ satisfies

$$
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Kripke Models for GK^{F}

A *Kripke model for* GK^{F} is a triple $\langle W, R, V \rangle$ such that $W \neq \emptyset$, $R : W^2 \to [0,1]$, and $V : Fm \times W \to [0,1]$ defined as for GK but:

$$
\begin{aligned}
V(\Box A, w) &= \inf_{u \in W}(Rwu \to_{\mathrm{G}} V(A, u)); \\
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\end{aligned}
$$

A formula φ is said to be GK-*valid* (GK^{F}-*valid*) if $V(\varphi, w) = 1$ for all Kripke models $\langle W, R, V \rangle$ for GK (GK^{F}) and $w \in W$.

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Known Results

- The fragments $GK_{\Box}$ and $GK^{F}_{\Box}$ coincide and their algebras are integral idempotent BPCRLs plus a unary operator $\Box$ satisfying:

$$\Box(x \wedge y) \approx \Box x \wedge \Box y; \qquad \mathrm{e} \approx \Box \mathrm{e}; \qquad \neg\neg\Box x \leq \Box\neg\neg x.$$

- The fragment $GK^{F}_{\Diamond}$ has the *finite model property* (and so is decidable) but the fragments $GK_{\Diamond}$ and $GK_{\Box}$ (or $GK^{F}_{\Box}$) do not.

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A Hypersequent Calculus for $GK_\Box$

A hypersequent calculus for $GK_\Box$ admitting cut-elimination is obtained by extending the calculus GG for Gödel logic with the rule:

$$\frac{\Pi \Rightarrow \mid \Gamma \Rightarrow \varphi}{\Box\Pi \Rightarrow \mid \Box\Gamma \Rightarrow \Box\varphi}$$

Theorem

$GK_\Box$ *is decidable (in fact, PSPACE-complete).*

Similar results holds for $GK_\Diamond$ and $GK^F_\Diamond$, but decidability for the full logics GK and GK^F is open.

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Non-Standard Calculi

- We have not yet defined calculi for some of the most famous fuzzy logics – *Łukasiewicz logic*, *product logic*, and *basic logic*.
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Łukasiewicz Logic

An Ł-*valuation* is a function v from formulas (in a fully expressive language based on $\to$ and $\bot$) to $[0,1]$ satisfying

$$v(\bot) = 0 \quad \text{and} \quad v(\varphi \to \psi) = \min(1, 1 - v(\varphi) + v(\psi)).$$

A formula φ is Ł-*valid*, written $\models_{Ł} \varphi$, if $v(\varphi) = 1$ for all Ł-valuations v.

(Or we could consider the BPCRL

$$\langle [0,1], \min, \max, *_{Ł}, \to_{Ł}, 1, 0, 0, 1 \rangle$$

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Interpreting Hypersequents for Łukasiewicz Logic

For a finite multiset of formulas Γ and valuation v, we define

$$\star^v_{Ł}(\Gamma) = 1 + \sum[v(\varphi) - 1 \mid \varphi \in \Gamma]$$

and interpret multiple-conclusion hypersequents by

$$\models_{Ł} \mathcal{G} \quad \text{iff} \quad \text{for all valuations } v\text{: } \star^v_{Ł}(\Gamma) \leq \star^v_{Ł}(\Delta) \text{ for some } (\Gamma \Rightarrow \Delta) \in \mathcal{G}.$$

However, note that for a *single-conclusion hypersequent* $\mathcal{G}$:

$$\models_{Ł} \mathcal{G} \quad \text{iff} \quad \models_{Ł} \mathcal{I}(\mathcal{G}).$$

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The Hypersequent Calculus GŁ

Axioms

$$\frac{}{\mathcal{G} \mid \varphi \Rightarrow \varphi}\ (\text{ID}) \qquad \frac{}{\mathcal{G} \mid \Rightarrow}\ (\text{EMP}) \qquad \frac{}{\mathcal{G} \mid \Gamma, \bot \Rightarrow \varphi}\ (\bot\Rightarrow)_Ł$$

Structural Rules:

$$\frac{\mathcal{G}}{\mathcal{G} \mid \mathcal{H}}\ (\text{EW}) \qquad \frac{\mathcal{G} \mid \mathcal{H} \mid \mathcal{H}}{\mathcal{G} \mid \mathcal{H}}\ (\text{EC}) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Delta}\ (\text{WL})$$

$$\frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ (\text{SPLIT}) \qquad \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\text{MIX})$$

Logical Rules

$$\frac{\mathcal{G} \mid \Gamma, \psi \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow)_Ł \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\Rightarrow\to)_Ł$$

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Example

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y \Rightarrow y}\,(\text{ID}) \quad \dfrac{}{x \Rightarrow x}\,(\text{ID})}{y, x \Rightarrow x, y}\,(\text{MIX})}{y, y \rightarrow x \Rightarrow x}\,(\rightarrow\Rightarrow)_{Ł} \qquad \dfrac{\dfrac{\dfrac{}{y \Rightarrow y}\,(\text{ID}) \quad \dfrac{}{x \Rightarrow x}\,(\text{ID})}{y, x \Rightarrow x, y}\,(\text{MIX})}{y, y \rightarrow x, x \Rightarrow x, y}\,(\text{WL})}{\dfrac{y, y \rightarrow x \Rightarrow x, x \rightarrow y}{(x \rightarrow y) \rightarrow y, y \rightarrow x \Rightarrow x}\,(\rightarrow\Rightarrow)_{Ł}}\,(\Rightarrow\rightarrow)_{Ł}$$

Example

$$\dfrac{\overline{y \Rightarrow y}\ (\text{ID}) \qquad \overline{x \Rightarrow x}\ (\text{ID})}{y, x \Rightarrow x, y}\ (\text{MIX}) \qquad \dfrac{\overline{y \Rightarrow y}\ (\text{ID}) \qquad \overline{x \Rightarrow x}\ (\text{ID})}{y, x \Rightarrow x, y}\ (\text{MIX})$$

$$\dfrac{y, x \Rightarrow x, y}{y, y \to x \Rightarrow x}\ (\to\Rightarrow)_{Ł} \qquad \dfrac{y, x \Rightarrow x, y}{y, y \to x, x \Rightarrow x, y}\ (\text{WL})$$

$$\dfrac{\dfrac{y, y \to x \Rightarrow x \qquad y, y \to x, x \Rightarrow x, y}{y, y \to x \Rightarrow x, x \to y}\ (\Rightarrow\to)_{Ł}}{(x \to y) \to y, y \to x \Rightarrow x}\ (\to\Rightarrow)_{Ł}$$

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$$
\dfrac{\dfrac{\dfrac{\overline{y \Rightarrow y}\,(\text{ID}) \quad \overline{x \Rightarrow x}\,(\text{ID})}{y, x \Rightarrow x, y}\,(\text{MIX})}{y, y \to x \Rightarrow x}\,(\to\Rightarrow)_Ł \qquad \dfrac{\dfrac{\overline{y \Rightarrow y}\,(\text{ID}) \quad \overline{x \Rightarrow x}\,(\text{ID})}{y, x \Rightarrow x, y}\,(\text{MIX})}{y, y \to x, x \Rightarrow x, y}\,(\text{WL})}{\dfrac{y, y \to x \Rightarrow x, x \to y}{(x \to y) \to y, y \to x \Rightarrow x}\,(\to\Rightarrow)_Ł}\,(\Rightarrow\to)_Ł
$$

Example

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{y \Rightarrow y}\,(\text{ID})
      \qquad
      \dfrac{}{x \Rightarrow x}\,(\text{ID})
    }{y, x \Rightarrow x, y}\,(\text{MIX})
  }{y, y \to x \Rightarrow x}\,(\to\Rightarrow)_{Ł}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{y \Rightarrow y}\,(\text{ID})
      \qquad
      \dfrac{}{x \Rightarrow x}\,(\text{ID})
    }{y, x \Rightarrow x, y}\,(\text{MIX})
  }{y, y \to x, x \Rightarrow x, y}\,(\text{WL})
}{
  \dfrac{y, y \to x \Rightarrow x, x \to y}{(x \to y) \to y, y \to x \Rightarrow x}\,(\to\Rightarrow)_{Ł}
}\,(\Rightarrow\to)_{Ł}
$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{y \Rightarrow y}\,(\text{ID}) \quad \dfrac{}{x \Rightarrow x}\,(\text{ID})
}{y, x \Rightarrow x, y}\,(\text{MIX})
}{y, y \to x \Rightarrow x}\,(\to\Rightarrow)_{Ł}
\qquad
\dfrac{
\dfrac{
\dfrac{}{y \Rightarrow y}\,(\text{ID}) \quad \dfrac{}{x \Rightarrow x}\,(\text{ID})
}{y, x \Rightarrow x, y}\,(\text{MIX})
}{y, y \to x, x \Rightarrow x, y}\,(\text{WL})
}{
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}\,(\Rightarrow\to)_{Ł}
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$$

Example

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(ID)}
\UnaryInfC{$y \Rightarrow y$}
\AxiomC{}
\RightLabel{(ID)}
\UnaryInfC{$x \Rightarrow x$}
\RightLabel{(MIX)}
\BinaryInfC{$y, x \Rightarrow x, y$}
\RightLabel{$(\rightarrow\Rightarrow)_{Ł}$}
\UnaryInfC{$y, y \rightarrow x \Rightarrow x$}
\AxiomC{}
\RightLabel{(ID)}
\UnaryInfC{$y \Rightarrow y$}
\AxiomC{}
\RightLabel{(ID)}
\UnaryInfC{$x \Rightarrow x$}
\RightLabel{(MIX)}
\BinaryInfC{$y, x \Rightarrow x, y$}
\RightLabel{(WL)}
\UnaryInfC{$y, y \rightarrow x, x \Rightarrow x, y$}
\RightLabel{$(\Rightarrow\rightarrow)_{Ł}$}
\BinaryInfC{$y, y \rightarrow x \Rightarrow x, x \rightarrow y$}
\RightLabel{$(\rightarrow\Rightarrow)_{Ł}$}
\UnaryInfC{$(x \rightarrow y) \rightarrow y, y \rightarrow x \Rightarrow x$}
\end{prooftree}
$$

Example

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{y \Rightarrow y}\,(\text{ID}) \qquad \dfrac{}{x \Rightarrow x}\,(\text{ID})
}{y, x \Rightarrow x, y}\,(\text{MIX})
}{y, y \to x \Rightarrow x}\,(\to\Rightarrow)_{Ł}
\qquad
\dfrac{
\dfrac{
\dfrac{}{y \Rightarrow y}\,(\text{ID}) \qquad \dfrac{}{x \Rightarrow x}\,(\text{ID})
}{y, x \Rightarrow x, y}\,(\text{MIX})
}{y, y \to x, x \Rightarrow x, y}\,(\text{WL})
}{
\dfrac{y, y \to x \Rightarrow x, x \to y}{(x \to y) \to y, y \to x \Rightarrow x}\,(\to\Rightarrow)_{Ł}
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$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{y \Rightarrow y}\,(\textsc{id})
      \qquad
      \dfrac{}{x \Rightarrow x}\,(\textsc{id})
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  }{y, y \to x \Rightarrow x}\,(\to\Rightarrow)_{\text{Ł}}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{y \Rightarrow y}\,(\textsc{id})
      \qquad
      \dfrac{}{x \Rightarrow x}\,(\textsc{id})
    }{y, x \Rightarrow x, y}\,(\textsc{mix})
  }{y, y \to x, x \Rightarrow x, y}\,(\textsc{wl})
}{
  \dfrac{y, y \to x \Rightarrow x, x \to y}{(x \to y) \to y, y \to x \Rightarrow x}\,(\to\Rightarrow)_{\text{Ł}}
}\,(\Rightarrow\to)_{\text{Ł}}
$$

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$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{y \Rightarrow y}\,(\text{ID}) \quad \dfrac{}{x \Rightarrow x}\,(\text{ID})
    }{
      y, x \Rightarrow x, y
    }\,(\text{MIX})
  }{
    y, y \rightarrow x \Rightarrow x
  }\,(\rightarrow\Rightarrow)_{Ł}
  \quad
  \dfrac{
    \dfrac{
      \dfrac{}{y \Rightarrow y}\,(\text{ID}) \quad \dfrac{}{x \Rightarrow x}\,(\text{ID})
    }{
      y, x \Rightarrow x, y
    }\,(\text{MIX})
  }{
    y, y \rightarrow x, x \Rightarrow x, y
  }\,(\text{WL})
}{
  \dfrac{
    y, y \rightarrow x \Rightarrow x, x \rightarrow y
  }{
    (x \rightarrow y) \rightarrow y, y \rightarrow x \Rightarrow x
  }\,(\rightarrow\Rightarrow)_{Ł}
}\,(\Rightarrow\rightarrow)_{Ł}
$$

Comments

- We can prove soundness and completeness for the (cut-free) calculus GŁ *semantically*; i.e.

$$\vdash_{\text{GŁ}} \mathcal{G} \quad \text{iff} \quad \models_{\text{Ł}} \mathcal{G}.$$

  We can also prove cut-elimination for GŁ + (CUT).

  G. Metcalfe, N. Olivetti, and D. Gabbay. Sequent and hypersequent calculi for abelian and Łukasiewicz logics. *ACM Transactions on Computational Logic*, 6(3):578–613, 2005.

- GŁ∀ + (CUT) is sound and complete with respect to MV-algebras but does not have cut-elimination. However, φ is valid in the standard [0, 1] MV-algebra iff $\vdash_{\text{GŁ}\forall} \bot \Rightarrow \underbrace{\varphi, \ldots, \varphi}_{n}$ for $n = 2, 3, \ldots$.

  M. Baaz and G. Metcalfe. Herbrand's Theorem, Skolemization, and Proof Systems for Łukasiewicz Logic. *Journal of Logic and Computation* 20 (2010), 35–54.

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- GŁ$\forall$ + (CUT) is sound and complete with respect to MV-algebras but does not have cut-elimination. However, φ is valid in the standard $[0,1]$ MV-algebra iff $\vdash_{\text{GŁ}\forall} \bot \Rightarrow \underbrace{\varphi, \ldots, \varphi}_{n}$ for $n = 2, 3, \ldots$.

  M. Baaz and G. Metcalfe. Herbrand's Theorem, Skolemization, and Proof Systems for Łukasiewicz Logic. *Journal of Logic and Computation* 20 (2010), 35–54.

Interpreting GŁ: Giles's Game

- Think of a sequent $\Gamma \Rightarrow \Delta$ for Łukasiewicz logic as representing statements made by *you* (Γ) and *me* (Δ) where in the atomic case, I agree to pay you \$1 for each false statement and vice versa.
- For every *run* of the game, a fixed *risk value* $\langle x \rangle \in [0, 1]$ is associated with x, where $\langle \bot \rangle = 1$ and $\langle \Gamma \rangle = \sum_{a \in \Gamma} \langle a \rangle$ (Γ atomic).
- To deal with implication, we adopt the rule:

  If I assert $\varphi \to \psi$, then whenever you choose to attack this statement by asserting φ, I must assert also ψ. (And vice versa, i.e., for the roles of me and you switched.)

- I *win* a run of the game beginning with $\Gamma \Rightarrow \Delta$ and ending with $a_1, \ldots, a_m \Rightarrow b_1, \ldots, b_n$ if $\langle a_1, \ldots, a_m \rangle \geq \langle b_1, \ldots, b_n \rangle$.

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Abstracting from particular risk assignments, we can also consider *disjunctive (winning) strategies* represented by derivations in a suitable hypersequent calculus.

Then a formula φ is Ł-valid iff I have a winning disjunctive strategy for $\Rightarrow \varphi$ iff $\Rightarrow \varphi$ is derivable in GŁ.

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Product Logic(s)

A calculus GCHL for *cancellative hoop logic* in a language with $\to$ and $\cdot$ is obtained from GŁ by removing the rule $(\bot\Rightarrow)_{Ł}$ and adding

$$\frac{\mathcal{G} \mid \Gamma, \varphi, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \cdot \psi \Rightarrow \Delta}\ (\cdot\Rightarrow) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \cdot \psi, \Delta}\ (\Rightarrow\cdot)_{P}$$

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These calculi can be used to prove standard completeness, decidability, and complexity results.

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Basic Logic

Unfortunately, following results of Ciabattoni et al., we cannot hope to find a “standard” hypersequent calculus for Hájek’s basic logic BL.

Perhaps we need more complicated structures such as *relational hypersequents* of the form

$$\Gamma_1 \lhd_1 \Delta_1 \mid \ldots \mid \Gamma_n \lhd_n \Delta_n \qquad \text{where each } \lhd_i \in \{<, \leq\} \text{ ?}$$

Indeed, a relational hypersequent calculus has been given for the *product-free fragment* of BL in

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Other Open Problems

- Find calculi for “cancellative” (fuzzy) logics such as PMTL.
- Tackle decidability and complexity issues for UL, MTL, etc.
- Prove standard completeness (density elimination?) for IUL.
- Develop proof-theoretic methods for (fragments of) first-order fuzzy logics suitable, e.g., for fuzzy description logics.

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Conclusions

- Hypersequent calculi (admitting cut elimination) provide an elegant algorithmic framework for presenting fuzzy logics.
- These calculi can be a useful tool for tackling problems such as decidability, complexity, interpolation, admissibility of rules, standard completeness, first-order extensions, etc.
- Further topics include other proof frameworks, non-commutative logics, propositional quantifiers, finite-valued logics, ...

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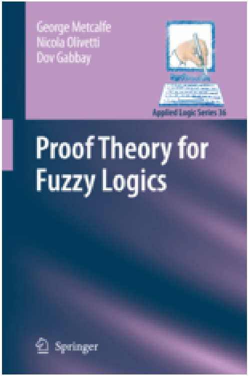
A Final Reference



G. Metcalfe, N. Olivetti, and D. Gabbay. *Proof Theory for Fuzzy Logics*. Volume 36 of Applied Logic. Springer, 2008.