n-contractive BL-logics

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§**1. Introduction**. The logic BL has been introduced by Hàjek in his book [H98] both as a common fragment of the most important fuzzy logics (Łukasiewicz, Gödel and product) and as the logic of continuous t-norms.

Recall that a t-norm is a binary map from $[0,1]$ into $[0,1]$, which is commutative, associative, isotonic and has 1 as neutral element. A continuous t-norm is a t-norm which is continuous.

A continuous t-norm $*$ has a residuum $\rightarrow_*$ defined by $x \rightarrow_* y = \sup\{z : z * x \leq y\}$.

If we interpret conjunction as $*$, implication as $\rightarrow_*$, $\wedge$ as min and $\vee$ as max, we obtain a logic, denoted by L_*, consisting of all formulas which are valid under this interpretation. The intersection of all L_* is said to be the logic of continuous t-norms.

It was proved (by Cignoli, Esteva, Godo and Torrens [CEGT], following some ideas of Hájek [H2]) that this logic coincides with the logic BL axiomatized by the following axioms:

(1) $\phi \to (\psi \to \phi)$.
(2) $(\phi \to \psi) \to ((\psi \to \gamma) \to (\phi \to \gamma))$.
(3) $(\phi \cdot \psi) \to (\psi \cdot \phi)$.
(4) $(\phi \cdot \psi) \to \phi$.
(5) $(\phi \to (\psi \to \gamma)) \to ((\phi \cdot \psi) \to \gamma)$.
(6) $((\phi \cdot \psi) \to \gamma) \to (\phi \to (\psi \to \gamma))$.
(7) $(\phi \cdot (\phi \to \psi)) \to (\psi \cdot (\psi \to \phi))$.
(8) $((\phi \to \psi) \to \gamma) \to (((\psi \to \phi) \to \gamma) \to \gamma)$.
(9) $0 \to \phi$,
and by the rule Modus Ponens: $\frac{\phi \quad \phi \to \psi}{\psi}$.

The connectives $\wedge, \vee, \neg, \leftrightarrow$ and the constant 1 are defined by:

$\phi \wedge \psi = \phi \cdot (\phi \to \psi)$,

$\phi \vee \psi = ((\phi \to \psi) \to \psi) \wedge ((\psi \to \phi) \to \phi)$,

$\neg\phi = \phi \to 0$,

$\phi \leftrightarrow \psi = (\phi \to \psi) \cdot (\psi \to \phi)$ and

$1 = 0 \to 0$.

The continuity of the t-norm is reflected by the divisibility axiom

$(\phi \wedge \psi) \to (\phi \cdot (\phi \to \psi))$.

Although there are even weaker many valued logics (for instance, Esteva and Godo's MTL), BL constitutes probably the first attempt to find a a really general fuzzy logic, and hence it is a fundamental contribution by Petr Hájek to this field.

BL is an algebraizable logic in the sense of [BP], and its equivalent algebraic semantics is the class of BL-algebras. These are commutative, integral, divisible, representable and bounded residuated lattices. Divisible means that if $a \leq b$ then there is a c such that $c \cdot b = a$. Representable means decomposable as a subdirect product of totally ordered algebras.

BL-algebras form a variety which is generated by the class of BL-chains. Moreover, every chain can be decomposed as ordinal sum of MV-chains (BL-chains with an involutive negation) and negative cones of ordered abelian groups with implication defined by $x \to y = (y - x) \wedge 0$, [AM]. More precisely:

Definition. Let $\langle I, \leq \rangle$ be a totally ordered set with minimum i_0. For all $i \in I$ let $\mathbf{A}_i$ be either an MV-chain or, if $i \neq i_0$, the negative cone of an ordered group. Assume that for $i \neq j$, $A_i \cap A_j = \{1\}$. Then $\bigoplus_{i \in I} \mathbf{A}_i$ (the *ordinal sum* of the family $(\mathbf{A}_i)_{i \in I}$) is the structure whose base set is $\bigcup_{i \in I} A_i$, whose bottom element is $\min(\mathbf{A}_{i_0})$, and whose operations are

$$x \to y = \begin{cases} x \to^{\mathbf{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } x \in A_i \text{ and } y \in A_j \text{ with } i > j \\ 1 & \text{if } x \in A_i \setminus \{1\} \text{ and } y \in A_j \text{ with } i < j \end{cases}$$

$$x \cdot y = \begin{cases} x \cdot^{\mathbf{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } x \in A_i \text{ and } y \in A_j \setminus \{1\} \text{ with } i > j \\ x & \text{if } x \in A_i \setminus \{1\} \text{ and } y \in A_j \text{ with } i < j \end{cases}$$

Note that ordinal sums in [AM] are slightly different from ordinal sums of t-norms. The main differences are: (a) here, there is always a first component, (b) components need not have a minimum and (c) the top element is common to all components.

§2. n-contractive BL-logics.

Contraction is a structural rule which can be expressed as an axiom:

$$\phi \to (\phi \cdot \phi).$$

Although BL is a contraction-free logic, divisibility is implied by (but it is not equivalent to) contraction, and hence it may be regarded as a weak form of contraction. Other weak forms of contraction are:

(1) The axiom (S) of strict negation, $\neg(\neg\phi \wedge \neg\neg\phi)$, which can also be regarded as a form of contraction, because it is equivalent to $\neg\phi \rightarrow (\neg\phi \cdot \neg\phi)$, i.e., contraction for negated formulas. The equivalent algebraic semantics is the variety of SBL-algebras, whose chains have a two element first component.

(2) The schema $\phi^n \rightarrow \phi^{n+1}$, which will be called n-contraction and will denoted by C_n.

In the sequel, BL plus C_n will be denoted by C_nBL.

The equivalent algebraic semantics of C_nBL is constituted by all n-potent BL-algebras, ie., by all BL-algebras satisfying the equation $x^{n+1} = x^n$.

Note that n-potent BL-chains are ordinal sums of MV-chains with cardinality $\leq n+1$. The MV-chain with $n+1$ elements will be denoted by $\mathbf{L}_n$.

Definition. Given any propositional fuzzy logic L, C_nL will denote the extension of L with the axiom schema C_n. Extensions of fuzzy logics by the schema C_n, and C_nMTL in particular, have been investigated first in [CEG]. Here is an important property of the logics C_nL, L any axiomatic extension of MTL:

(1) Any schematic extension L of C_nL has the global deduction theorem: $\phi \vdash_L \psi$ iff $\vdash_L \phi^n \to \psi$.

In my presentation in Prague, I was led to the following WRONG conclusion:

Hence, every subquasivariety of the class of C_nL-algebras is a variety.

As pointed out to me by James Raftery, this conclusion is wrong, even in the case of quasivarieties of C_nBL-algebras. Here is a counterexample:

Consider the quasivariety $\mathcal{V}$ generated by $\mathbf{L}_2 \times \mathbf{L}_1$. Clearly, $\mathcal{V}$ is a quasivariety of C_2BL-algebras.

One may check that there is no element x in $\mathbf{L}_2 \times \mathbf{L}_1$ such that $x^2 = 0$ and $\neg x \leq x$. Hence, the quasiequation $\gamma : (x^2 = 0 \& \neg x \leq x) \Rightarrow 0 = 1$ holds in $\mathcal{V}$.

But it is easy to check that γ does not hold in $\mathbf{L}_2$.

Since $\mathbf{L}_2$ is a homomorphic image of an element of $\mathcal{V}$, namely $\mathbf{L}_2 \times \mathbf{L}_1$, we conclude that $\mathcal{V}$ is not closed under homomorphic images and hence it is not a variety.

Note that the previous argument shows that there are quasivarieties of C_nBL-algebras which are not generated by chains. So the problem arises to characterize the quasivarieties of commutative and integral residuated lattices which are generated by chains. The previous example shows that prelinearity is not enough.

If L has the finite model property (i.e., every finite consequence relation not valid in L can be invalidated in a finite model of L), then we have:

L $\vdash \phi$ iff for every n, C_nL$\vdash \phi$.

The axiomatic extensions of C_nBL have other interesting properties. One of them is that any subvariety of the variety of n-potent BL-algebras is locally finite, i.e., any finitely generated subalgebra of an algebra from the variety is finite.

This is in fact a strengthening of the finite model property.

Another interesting fact about n-contraction in BL is the following: let psBL be obtained from BL by deleting commutativity of $\cdot$. Then psBL plus C_n coincides with C_nBL. That is, in psBL, bounded contraction implies exchange.

In this talk, for every $n > 1$, we will investigate the following extensions of C_nBL:

(a) C_nBL itself.

(b) DC_nBL, that is, C_nBL plus

(D_n) $(\phi^{m-1} \leftrightarrow (\phi \to \phi^n)) \to \phi^n$

for all m which does not divide n.

[The algebraic meaning of D_n is based on the following fact: the totally ordered models of DC_nBL are the ordinal sums of MV-algebras of cardinality k such that $k-1$ divides $n-1$, and not just of MV-chains with cardinality $\leq n+1$. We will see that this property is important for interpolation].

(c) SC_nBL and SDC_nBL, that is, C_nBL (DC_nBL respectively) plus the strong negation axiom S.

Most of the results presented in this talk have been obtained in two papers in collaboration with Matteo Bianchi [BM1] and [BM2].

§3. Decision problems, countermodels.

As a consequence of local finiteness, for every n, C_nBL has the finite model property. This result can be strengthened:

Theorem. For every n, k, there are BL-chains $\mathbf{C}^i_{n,k}$ $(i = 1, \ldots, n)$ with cardinality bounded by $(k+1)^{\frac{n(n+1)}{2}}$ such that a formula ϕ with k variables is provable in C_nBL iff it is true in $\mathbf{C}^i_{n,k}$ for $i = 1, \ldots, n$.

For DC_nBL, SC_nBL and for SDC_nBL, we have a stronger result, that is, the finitely many chains can be replaced by one single chain with an even better bound for its cardinality.

Hence, tautologicity in C_nBL, SC_nBL, DC_nBL ans SDC_nBL is co-NP-complete, and 1-satisfiability is NP-complete.

§4. **Strong completeness and universal chains**. Since fuzzy logics can also be regarded as logics of chains, an interesting question for any fuzzy logic L is the following: is there an L-chain $\mathbf{A}$ such that L is strongly complete with respect to $\mathbf{A}$? As shown in [CEGGMN], this question is equivalent to: is there an L-chain in which every countable L chain embeds?

The answer is YES for SC_nBL, DC_nBL and SDC_nBL and it is NO for C_nBL with $n > 1$.

For DC_nBL it suffices to take an ordinal sum, indexed by the non-negative rationals, of copies of $\mathbf{L}_n$.

For C_nBL, the problem is with the first component: if $p < n$ is coprime with n, then no ordinal sum beginning by $\mathbf{L}_n$ can embed into an ordinal sum beginning by $\mathbf{L}_p$ and viceversa.

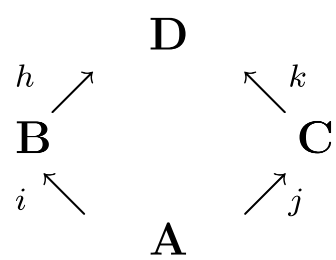
§5. Interpolation and amalgamation.

Definition. A logic L has the deductive interpolation property if for any set Γ of formulas and for any formula ψ, if $\Gamma \vdash_L \psi$, then there is a formula γ whose variables are common to Γ and ψ, such that $\Gamma \vdash_L \gamma$ and $\gamma \vdash_L \psi$.

This means that only a part of the information contained in Γ is necessary to derive ψ, and this part of information can be expressed using only the variables common to Γ and to ψ.

For algebraizable logics whose equivalent algebraic semantics is a variety of pointed and commutative residuated lattices, deductive interpolation is equivalent to the amalgamation property for its correponding variety, [GJKO].

Definition. A variety $\mathcal{V}$ has the amalgamation property, if given $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C} \in \mathcal{V}$ and given two embeddings i and j from $\mathbf{A}$ into $\mathbf{B}$ and $\mathbf{C}$, respectively, there are $\mathbf{D} \in \mathcal{V}$ and embeddings h and k from $\mathbf{B}$ and from $\mathbf{C}$, respectively, into $\mathbf{D}$, such that the diagram



commutes.

This close relationship between interpolation and amalgamation is an important success of Abstract Algebraic Logic, because it relates a purely logical concept to a purely algebraic one.

In [Mo06] it is shown that the variety of BL- algebras has the amalgamation property, and hence BL has the deductive interpolation property.

Somehow surprisingly, bounded contraction destroys the amalgamation property, and hence the interpolation property. Consider for instance the variety of 3-potent BL-algebras.

Then, $\mathbf{L}_1$, $\mathbf{L}_2$ and $\mathbf{L}_3$ are 3-potent and $\mathbf{L}_1$ is a subalgebra of the other two. But an amalgam of these algebras should contain as a subalgebra $\mathbf{L}_{\mathrm{gcm}\{2,3\}} = \mathbf{L}_6$, which is not in the variety.

To the contrary, the varieties corresponding to DC_nBL and SDC_nBL have the amalgamation property, and hence the corresponing logics have deductive intepolation. The idea is that one can reduce amalgamation of DC_nBL-algebras to amalgamation of their MV-components, and any three MV-components of a DC_nBL chain have a common amalgam into $\mathbf{L}_n$.

Hence, the axiom D_n is crucial in order to restore deductive interpolation. It would be interesting to understand the proof theoretic meaning of this axiom (if any!).

§**6. Completions**. In Hájek's semantics for first order fuzzy logics, the existential quantifiers and the universal quantifiers are interpreted by means of suprema and infima respectively. This forces us to consider only safe interpretations, that is, interpretations in which the required suprema and infima exist.

Of course for complete L chains, there is no problem about safety. Hence, one may wonder what happens if we restrict our semantics to complete L-chains.

In general, we have problems, because complete L-chains are not general enough. In particular, sometimes it is not possible to embed an L-chain into a complete L-chain in which existing suprema and infima are preserved, and sometimes it is no possible to embed an L-chain into a complete L-chain at all.

As a start, it is interesting to investigate for which varieties $\mathcal{V}$ of BL-algebras, every chain in $\mathcal{V}$ has a completion which is still in $\mathcal{V}$.

Definition. A variety $\mathcal{V}$ of BL-algebras is closed under completions if every algebra in $\mathcal{V}$ embeds into a complete algebra in $\mathcal{V}$.

Note that, for varieties $\mathcal{V}$ of representable residuated lattices, closure under completions is equivalent to the weaker condition:

Every chain in $\mathcal{V}$ embeds into a complete algebra in $\mathcal{V}$.

There are several ways to embed a (commutative) residuated lattice into a complete one. The most important ones are the MacNeille completion, the canonical completion and the dual canonical completion.

The MacNeille completion has the remarkable property that existing suprema and infima are preserved when passing from a residuated lattice to its MacNeille completion.

However, MacNeille completion works badly for arbitrary BL-algebras: even the MacNeille completion of a Gödel algebra need not be a Gödel algebra. However, in some cases it works well for BL-chains.

In any case, Busanice and Cabrer were able to give a very informative picture of the situation for BL-algebras.

Theorem. (1) A variety $\mathcal{V}$ of BL-algebras is closed under canonical completions iff it is finitely generated.

(2) Let $\mathcal{V}$ be a variety of BL-algebras. The following are equivalent:

(2a) $\mathcal{V}$ closed under completions.

(2b) $\mathcal{V}$ is closed under dual canonical completions.

(2c) $\mathcal{V}$ is a variety of n-potent BL-algebras for some n.

The proof of the second part divides into three steps:

(1) It can be proved that a variety of BL-algebras which contains either an infinite product chain or an infinite MV-chain is not closed under completions.

(2) It can be proved that a variety of BL-algebras which contains no infinite product chain or MV-chain is a variety of n-potent BL-algebras. This is also proved in [BM].

(3) It can be proved that any chain in a variety of n-potent BL algebras embeds into a complete chain in the same variety (this is also proved in [BM]).

Indeed, represent the chain as an ordinal sum $\bigoplus_{i \in I} \mathbf{W}_i$. Take the (MacNeille or, alternatively, the canonical) completion J of the index set I, and consider the ordinal sum $\bigoplus_{j \in J} \mathbf{U}_j$, where $\mathbf{U}_j = \mathbf{W}_j$ is $j \in I$, and $\mathbf{U}_j = \mathbf{L}_2$ otherwise. This is a completion of the original algebra.

Moreover if J is the canonical completion of I, we obtain the dual canonical completion of the original algebra, and if J is the MacNeille completion of I, we obtain the MacNeille completion of the original algebra, and hence, original suprema and infima are preserved.

Finally, in both cases every equation which is valid in the original algebra remains valid in its complete extension defined in this way.

§**7. Complexity of the semantics by complete chains**. For BL and its first order extension, an important problem is the standard completeness, that is the completeness with respect to the class of BL-chains on $[0,1]$. Now there are no models on $[0,1]$ for C_nBL or for its extensions, with the exception of C_1BL which coincides with Gödel logic. Indeed, models for C_nBL which are not Gödel chains fail to be densely ordered.

Hence, if $n > 1$, none of the logics examined so far is sound and complete with respect to the standard semantics. However, it makes sense to investigate completeness of the first order version L∀ of any logic L$\in \{C_nBL, DC_nBL, SC_nBL, SDC_nBL\}$ with respect to the semantics given by complete chains.

Theorem. Let L be an axiomatic extension of BL. Then, L$\forall$ is complete with respect to the class of its complete chains iff it extends C_nBL.

The $\Leftarrow$ direction follows from the fact that the class of chains of a variety of n-potent BL-algebras is closed under MacNeille completions.

An interesting consequence is the following:

Corollary. If L is a recursively axiomatizable extension of BL, then the class of L$\forall$ tautologies for the class of complete L-chains is recursively enumerable iff L extends C_nBL for some n.

§**8. Supersoundness**. As we said before, the interpretation of first order formulas of a first order fuzzy logic L∀ is restricted to safe structures. We have already discussed the semantics based on complete chains, in which all suprema and infima, and not only those required by the interpretation, exist.

On the opposite side, one may wonder what happens if we also consider interpretations which are not safe, but in which the formula taken into consideration has a truth value. Tis leads to the following definition:

Definition. A formula ϕ of L∀ is called supersound if it is true in every (possibly unsafe) structure in which the truth value of ϕ is defined. A logic L∀ is said to be supersound if every theorem of L∀ is supersound.

The following result is easy to prove:

Theorem. If the MacNeille completion of every L-chainis still an L-chain, then L∀ is supersound.

Indeed every interpretation into an L-chain can be extended into a safe one.

Hence, if L is any schematic extension of C_nBL, then L∀ is supersound.

To the contrary, if L is a schematic extension of BL which does not extend C_nBL for any n, then it can be proved that L∀ is not supersound. Therefore:

Theorem [BM], a slight generalization of [HS]. Suppose L is a schematic extension of BL. Then, L$\forall$ is supersound iff L proves n-contraction for some n.

Thus n-contractiveness is equivalent to supersoundness, to closure under completions and to recursive axiomatizability of the formulas valid in all complete chains.

§**9. Conclusions** The logic BL constitutes a a very important contribution of Petr Hájek to Fuzzy Logic. Contractive BL-logics constitute good approximations of BL, as BL is the intersection of all C_nBL.

For n-contractive BL-logics it is easy to build countermodels, in that they only depend on the number of propositional variables.

The difference between BL and C_nBL is particularly relevant in first order fuzzy logics, because the class of chains in a variety of n-potent BL-algebras is closed under MacNeille completion, and hence one can work directly in complete chains.

Although bounded contraction is a quite natural structural rule, it destroys interpolation, and to restore it we need another axiom (namely, D_n), whose proof theoretic meaning is unclear.

DĚKUJI!!!

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