Mathematical Fuzzy Logic in Linguistic Semantics

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Outline

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Mathematical fuzzy logic

Main goals:

- Provide working mathematical model of the vagueness phenomenon. The mathematization is based on introduction of degrees of truth taken from an ordered scale.
- Offer more proper formal tools for modeling of human reasoning and, at least, some parts of linguistic semantics

MFL is established sound formal system; its applications are well justified

J. A. Goguen, J. Pavelka — *logic and algebra, the sixties and seventies*

U. Höhle, D. Mundici, A. DiNola — *algebra, the eighties and nineties*

V. Novák, P. Hájek, S. Gottwald — *logic, formal FST, the eighties and nineties*

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Mathematical fuzzy logic

Fuzzy logic narrow sense (FLn)

- propositional
- predicate — first order
- higher order — fuzzy type theory

Several kinds of FLn depending on the structure of truth values

Works of L. A. Zadeh on linguistic semantics

- A fuzzy-set-theoretic interpretation of linguistic hedges (J. Cyber. 1972)
- Quantitative fuzzy semantics (Inf. Sci. 1973)
- The concept of a linguistic variable and its application to approximate reasoning (inf. Sci. 1975)
- PRUF — a meaning representation language for natural languages (Int. J. Man-Mach. Stud. 1978)
- Test-score semantics for natural languages and meaning representation via PRUF (Empirical Semantics 1981)
- A computational approach to fuzzy quantifiers in natural languages (Comp. Math. with Applic. 1983)

Semantic space E_K

generated by a kernel space K
(*set of smells, plants, natural numbers*, etc.)

The meaning of a linguistic expression (syntagm)

$$\mathscr{A} \mapsto A \underset{\sim}{\subseteq} E_K$$

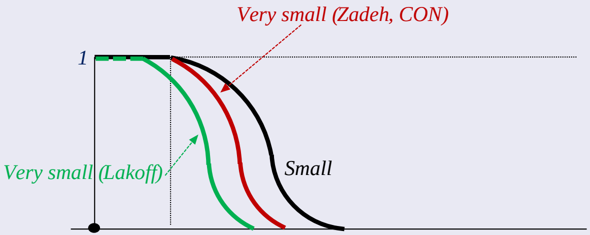
The meaning of more complicated expressions is constructed from simpler ones using operations on fuzzy sets

Linguistic hedges

Special adverbs (*very, more or less, slightly*, etc.)
Operations on fuzzy sets, e.g.

$$\mathrm{CON}(a) = a^2$$
$$\mathrm{DIL}(a) = a^{0.5}$$

Modification using hedges



Linguistic variable

Definition

$$\langle \mathscr{X}, T(\mathscr{X}), U, G, M \rangle$$

- $\mathscr{X}$ — name of the variable
  Age, height, pressure, size, etc.
- $T(\mathscr{X})$ — term set
  (*young, very young, more or less young, very very young, etc.*)
- U — universe
- G syntactic rule (context-free grammar)
- M — semantic rule

$$M(\mathscr{A}) = A \underset{\sim}{\subset} U$$

IRAFM

Fuzzy quantifier is a fuzzy number $Q \underset{\sim}{\subseteq} \mathbb{R}$ (truth value)

(i) First kind (*several, few, many*, etc.)
characterize cardinality of the fuzzy set

$$\Sigma\text{Count}(A) = \sum_{x \in U} A(x).$$

"There are $\mathscr{Q}A$'s" $= Q(\Sigma Count(A))$

(ii) Second kind (*most, a large fraction, much of*, etc.)
characterize relative cardinality of the fuzzy set

$$\Sigma Count(B|A) = \frac{\sum_{x \in U}(A \cap B)(x)}{\sum_{x \in U} A(x)}$$

"$\mathscr{Q}B$'s are A" $= Q(\Sigma Count(B|A))$

Other measures: FGCount, FECount

Quantifier **Many**: fuzzy quantifier of the first kind

Language of predicate fuzzy logic extended by $\int$
$\mathscr{M}$ — a finite model

$$\mathscr{M}(\int A dx) = \frac{1}{n} \sum_{d \in M} \mathscr{M}(A_x[\mathbf{d}])$$

Completeness with this kind of quantifier is preserved

Classical theory of generalized quantifiers

A. Mostowski, P. Lindström, J. van Benthem, J. Barwise, R. Cooper, L. E. Keenan, D. Westerståhl

Definition (Generalized quantifier Q of type $\langle k_1, \dots, k_n \rangle$)

Function assigning to each set M a relation

$$\mathbf{Q}_M \subseteq P(M^{k_1}) \times \cdots \times P(M^{k_n})$$

so that

$$\mathscr{M}((Q\mathbf{x}_1 \cdots \mathbf{x}_n)(A_1, \dots, A_n)) = \mathbf{1} \quad \text{iff} \\ \langle \mathrm{Sat}_{\mathscr{M}}(A_1(x_{11}, \dots, x_{1k_1})), \dots, \mathrm{Sat}_{\mathscr{M}}(A_1(x_{n1}, \dots, x_{nk_n})) \rangle \in \mathbf{Q}_D$$

Fuzzy logic: I. Glöckner (semi-fuzzy quantifiers),
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Fuzzy logic broader sense (FLb)

Initiated in 1995

Paradigm

(i) Develop a working mathematical model of linguistic semantics (at least some of its parts)

(ii) Develop a mathematical model of natural (commonsense) human reasoning

Special paradigm, extension of FLn not identical with L. A. Zadeh's concept of FL in wide sense!

Constituents of FLb:

- Theory of evaluative linguistic expressions
- Theory of intermediate quantifiers and generalized Aristotle's syllogisms
- Theory of fuzzy/linguistic IF-THEN rules and logical inference (Perception-based Logical Deduction)
- Formal model of special commonsense reasoning cases

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What can MFL offer to linguistic semantics?

MFL is first of all the logic

To apply its power in the development of a mathematical model of linguistic semantics, we must follow:

- Results of classical linguistics
- Results and methods of logical analysis of the semantics of natural language

Source of inspiration

- Montague grammar
- Logical analysis of concepts (P. Tichý, P. Materna)
  Transparent Intensional Logic

Cooperate with linguists and philosophers!

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Functional Generative Description of Natural language

P. Sgall, E. Hajičová and J. Panevová

FGD

Description of a sentence is understood as a sequence of its representations on certain ordered levels. The lowest level stresses the outer (sound) form of the sentence while the highest one represents its meaning.

Functional Generative Description of Natural language

Five levels

1. *phonetic* (PH),
2. *phonemic* (PM),
3. *morphemic* (MR),
4. *surface syntactic* (SS)
5. *tectogrammatical* TR.

The meaning of sentence on tectogrammatical level

Tectogrammatical dependency tree

Root represents verb, nodes represent word shapes, edges represent dependency relations.

Labels of nodes are word forms (complex units) that consist of elements of the following categories:

1. Lexical unit; the basic lexical units are *nouns, adjectives, adverbs* and *verbs*
2. Information about membership in the *topic* or *focus*
3. *Grammatemes* (morphological meanings) of several categories for each class of words

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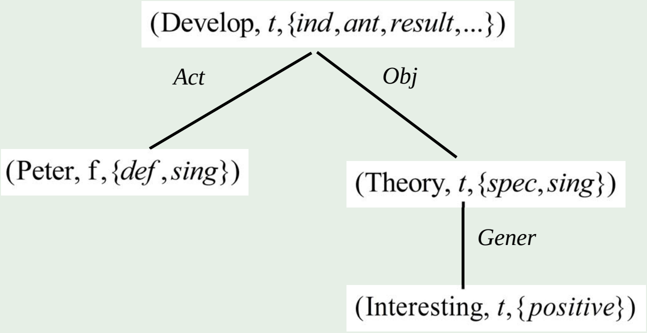
The meaning of sentence on tectogrammatical level

Example

PETER developed an interesting theory



Logical model of linguistic semantics

For logical analysis of linguistic semantics, it is necessary to consider the following concepts

(i) Possible world w, set of possible worlds W
(Maximal) *consistent set of facts*

(ii) Time
Branching chronologies

(iii) Intension

(iv) Extension

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Necessity of the concept of intension

Expressions of natural language are names of intensions and they cannot be identified with their extensions.

Frege's compositionality principle

If the meaning of a compound expression E depends exclusively on the meanings of its subexpressions, then substituting a subexpression E' by E'' that possesses the same meaning as E' cannot change the meaning of E.

Implies the principle of substitutivity

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Implies the principle of substitutivity

Necessity of the concept of intension

Identifying the meaning with extensions harms the principle of substitutivity:

- Morning star = $\{x_0\}$, Evening star = $\{x_1\}$

  Morning star *is* *evening star* (Quine)
  w_1 = Earth : $x_0 = x_1$ (Venus)
  w_2 = Jupiter?
- Settlement of pope = {Rome}
  Settlement of pope *was in France* ⇒ **Rome** *was in France*
- Colleagues of Paul = M_C
  Paul works in the University of Ostrava,
  i.e. M_C ='professors in UO"

  Former *colleagues of Paul* *play bridge*
  Former **professors in UO** *play bridge*

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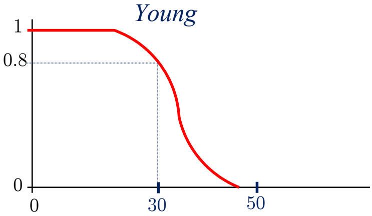
Young $\mapsto$ 

Berta is 30, i.e. she is 80% young

Berta is a dog!?

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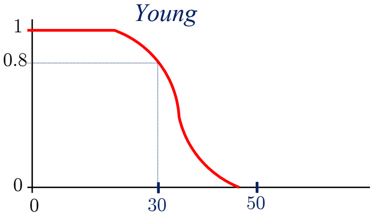

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$\mathscr{A}$ — a natural language expression
assign it a formula $A(x)$ of the language of predicate $Ev_Ł$

Intension of $\mathscr{A}$

$$\underline{A} = \{{}^{a_t}/A_x[t] \mid t \text{ closed terms}\}$$

t — *names* of objects having the property $A(x)$

Context, Extension — determined by a model $\mathscr{M}$

Models of intension and extension require higher-order logic

Intension in fuzzy logic with evaluated syntax $Ev_Ł$

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Generalization of classical **type theory**

- B. Russel
- A. Church, L. Henkin, P. Andrews, P. Martin-Löf

The syntax is an extended lambda calculus

- more logical axioms
- many-valued semantics

Higher order fuzzy logic

Generalization of classical **type theory**

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- more logical axioms
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Syntax

Types

Elementary types: o (truth values), ϵ (objects)
Composed types: $\beta\alpha$

Formulas have types: $A_\alpha \in Form_\alpha$, $A_\alpha \equiv B_\alpha$, $\lambda x_\alpha\, C_\beta$, $\mathbf{\Delta}_{oo}$

Formulas of type o are propositions

Important: fuzzy equality

$$A_\alpha \equiv B_\alpha$$

Interpretation of formulas $A_{\beta\alpha}$ are functions $M_\alpha \longrightarrow M_\beta$

Description operator $\iota_{\alpha(o\alpha)}$:

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IRAFM

Truth values in FTT

Truth values should form either of:

1. a complete linearly ordered IMTL_Δ-algebra
2. linearly ordered MV_Δ-algebra
3. linearly ordered BL_Δ-algebra
4. linearly ordered EQ_Δ-algebra or IEQ_Δ-algebra

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Truth values in FTT

Standard Łukasiewicz$_\Delta$-algebra

$$\mathscr{L} = \langle [0,1], \vee, \wedge, \otimes, \Delta, \rightarrow, 0, 1 \rangle$$

$$\vee, \wedge = \text{minimum, maximum}$$

$$\otimes = \text{left continuous t-norm,} \qquad a \otimes b = 0 \vee (a + b - 1)$$

$$\rightarrow = \text{residuation} \qquad a \rightarrow b = 1 \wedge (1 - a + b)$$

$$\neg a = a \rightarrow 0 \ (= 1 - a), \qquad \neg\neg a = a$$

$$\Delta(a) = \begin{cases} \mathbf{1} & \text{if } a = \mathbf{1}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Special definitions

(a) Representation of truth

$$\top := (\lambda x_o\, x_o \equiv \lambda x_o\, x_o)$$

(b) Representation of falsity

$$\bot := (\lambda x_o\, x_o \equiv \lambda x_o\, \top)$$

(c) Negation:

$$\neg := \lambda x_o(\bot \equiv x_o)$$

(d) Implication:

$$\Rightarrow\ := \lambda x_o \lambda y_o((x_o \wedge y_o) \equiv x_o)$$

(e) Special connectives: $\boldsymbol{\&}, \wedge, \vee, \boldsymbol{\Delta}$

IRAFM

Logical axioms of IMTL-FTT (IMTL$_\Delta$ algebra)

4 fundamental axioms

(FT1) $\mathbf{\Delta}(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv f_{\beta\alpha}\, y_\alpha)$

(FT3) $(\lambda x_\alpha B_\beta)A_\alpha \equiv C_\beta$

where C_β is obtained from B_β by replacing all free occurrences of x_α in it by A_α, provided that A_α is substitutable to B_β for x_α (*lambda conversion*).

2 equivalence axioms

(FT7) $(A_o \equiv \top) \equiv A_o$

4 implication axioms, 3 conjunction axioms, 3 delta axioms

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Logical axioms of IMTL-FTT (IMTL$_\Delta$ algebra)

4 fundamental axioms

(FT1) $\boldsymbol{\Delta}(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv f_{\beta\alpha}\, y_\alpha)$

(FT3) $(\lambda x_\alpha B_\beta)A_\alpha \equiv C_\beta$

where C_β is obtained from B_β by replacing all free occurrences of x_α in it by A_α, provided that A_α is substitutable to B_β for x_α (*lambda conversion*).

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Predicate axiom

(FT17) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha)B_o)$ $\quad$ x_α is not free in A_o

Axiom of descriptions

(FT18) $\iota_{\epsilon(o\epsilon)}(\mathbf{E}_{(o\epsilon)\epsilon}\, y_\epsilon) \equiv y_\epsilon$

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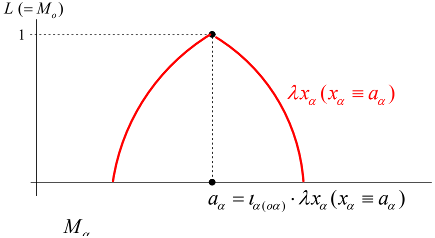

Inference rules

Rule (R) *Let $A_\alpha \equiv A'_\alpha$ and $B \in Form_o$. Then infer B' where B' comes form B by replacing one occurrence of A_α, which is not preceded by λ, by A'_α.*

Rule (N) *Let $A_o \in Form_o$ be a formula. Then from A_o infer $\mathbf{\Delta} A_o$.*

Theory

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- $T \vdash A$ — as usual

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Frame

$$\mathscr{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathscr{L}_\Delta \rangle$$

(i) $\mathscr{L}_\Delta$: Łukasiewicz Δ-algebra
(IMTL$_\Delta$-algebra, IEQ$_\Delta$-algebra, BL$_\Delta$-algebra)

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Semantics

(ii) Fuzzy equality $=_\alpha$: $M_\alpha \times M_\alpha \longrightarrow L$

$$[x =_\alpha x] = \mathbf{1} \qquad \text{(reflexivity)}$$

$$[x =_\alpha y] = [y =_\alpha x] \qquad \text{(symmetry)}$$

$$[x =_\alpha y] \otimes [y =_\alpha z] \leq [x =_\alpha z] \qquad \text{(transitivity)}$$

Example

$$[x =_\alpha y] = 0 \vee (1 - |x - y|)$$

$$[x =_\alpha y] = \begin{cases} \mathbf{1}, & \text{if } x = y, \\ a, & \text{otherwise} \end{cases}$$

$$[x =_\alpha y] = \begin{cases} 1, \text{if } x = y \\ \frac{1}{v-u} \cdot ((v - x) \wedge (v - y)) \vee ((x - u) \wedge (y - u)) \end{cases}$$

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Semantics — scheme of frame

$$(M_o = \{a \mid a \in L\}, \leftrightarrow) \qquad (M_\epsilon = \{u \mid \varphi(u)\}, =_\epsilon)$$

$$(M_{oo} \subseteq \{g_{oo} \mid g_{oo} : M_o \longrightarrow M_o\}, =_{oo})$$

$$(M_{o\epsilon} \subseteq \{f_{o\epsilon} \mid f_{o\epsilon} : M_\epsilon \longrightarrow M_o\}, =_{o\epsilon})$$

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Interpretation of formulas

$$\mathscr{M}^{\mathscr{E}}(A_{\beta\alpha}) \in M_{\beta\alpha}$$

Example (interpretation)

$\mathscr{M}^{\mathscr{E}}(A_o) \in E$ is a truth value

$\mathscr{M}^{\mathscr{E}}(A_\alpha \equiv B_\alpha) \in E$ is a truth degree of fuzzy equality between $\mathscr{M}^{\mathscr{E}}(A_\alpha)$ and $\mathscr{M}^{\mathscr{E}}(B_\alpha)$

$\mathscr{M}^{\mathscr{E}}(A_{o\epsilon}) : M_\alpha \longrightarrow E$ is a fuzzy set in M_ϵ

$\mathscr{M}^{\mathscr{E}}(A_{(o\epsilon)\epsilon}) : M_\epsilon \longrightarrow M_o^{M_\epsilon}$ is a fuzzy relation on M_ϵ

$\mathscr{M}^{\mathscr{E}}(A_{\epsilon\epsilon}) : M_\epsilon \longrightarrow M_\epsilon$ is a function on objects

Special formulas

$$\Upsilon_{oo} \equiv \lambda z_o \cdot \neg \mathbf{\Delta}(\neg z_o)$$

Semantics:

$$\mathscr{M}(\Upsilon z_o) = \mathbf{1} \quad \text{iff} \quad z_0 > \mathbf{0}$$

$$\hat{\Upsilon}_{oo} \equiv \lambda z_o \cdot \neg \mathbf{\Delta}(z_o \vee \neg z_o) \equiv \lambda z_o \cdot \neg \mathbf{\Delta} z_o \wedge \neg \mathbf{\Delta} \neg z_o)$$

Semantics (general truth value):

$$\mathscr{M}(\hat{\Upsilon} z_o) = \mathbf{1} \quad \text{iff} \quad \mathbf{1} > z_0 > \mathbf{0}$$

Theorem (Generalized completeness (Henkin style))

(a) *A theory T of fuzzy type theory is consistent iff it has a general model $\mathscr{M}$.*

(b) *For every theory T of FTT and a formula A_o*

$$T \vdash A_o \quad \text{iff} \quad T \models A_o.$$

FTT has a lot of interesting properties and great explication power

How the meaning of the following can be modeled?

The present king of France

- $f : X \longrightarrow Y$, $f(x')$ for some $x' \in X$ is undefined
- $f(x') = \sharp$ where $\sharp \notin Y$

Introducing partial functions in FTT

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FTT with partial functions

Adding special element to sets of given type and special constant:

(i) $\sharp_\epsilon$

(ii) $\sharp_{\beta\alpha} := \lambda x_\alpha \, \sharp_\beta \qquad \beta \neq o$

New axioms:

$$
\begin{aligned}
f_{\beta\alpha}\,\sharp &\equiv \sharp, && \beta \neq o \\
f_{o\alpha}\,\sharp &\equiv \bot \\
\sharp_\alpha &\equiv \sharp_\alpha, && \alpha \neq o \\
(\sharp_\alpha \equiv x_\alpha) &\equiv \bot, && x_\alpha \neq \sharp
\end{aligned}
$$

Additional special primitive types: ω — possible world, τ — time

- **Intension** $\lambda w_\omega \, \lambda t_\tau \, A_\alpha$

  Each possible world w_ω is assigned a chronology of objects of type α

- **Extension**: fixed possible world w_ω and time t_τ, $\alpha = o\beta$

$$(\lambda w_\omega \, \lambda t_\tau \, A_{o\beta}) w \, t$$

Interpretation is a fuzzy set $\mathscr{M}((\lambda \omega \, \lambda \tau \, A_{o\beta}) w \, t) \underset{\sim}{\subset} M_\beta$

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Outline

Evaluative linguistic expressions

Special expressions of natural language using which people evaluate phenomena surrounding them.

They are omnipresent in natural language (people permanently evaluate).

They occur in description of any process, decision situation, procedure, characterization of objects, etc.

Seminal works: L. A. Zadeh

Grammatical structure

(i) *Simple evaluative expressions*:

(a) $\langle$pure evaluative expressions$\rangle$:= $\langle$linguistic hedge$\rangle\langle$TE-adjective$\rangle$
(b) $\langle$fuzzy quantity$\rangle$:= $\langle$linguistic hedge$\rangle\langle$numeral$\rangle$

$\langle$numeral$\rangle$ – name of an element from the scale
$\langle$linguistic hedge$\rangle$:= empty | $\langle$narrowing adverb$\rangle$
| $\langle$widening adverb$\rangle$ | $\langle$specifying adverb$\rangle$

(ii) *Negative evaluative expressions*

not $\langle$pure evaluative expression$\rangle$

(iii) *Compound evaluative expressions*

(a) $\langle$pure evaluative expression$\rangle$ or $\langle$pure evaluative expression$\rangle$
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Grammatical structure

Example

(i) TE-adjectives: *small, medium, big; weak, medium strong, strong; silly, normal, intelligent*

(ii) Narrowing hedges: *very, extremely, significantly*

(iii) Widening hedges: *more or less, roughly, very roughly*

(iv) Specifying adverbs: *approximately, about, rather, precisely*

(v) Pure evaluative expressions: *very short, more or less strong, more or less medium, roughly big, about twenty five*

(vi) Fuzzy numbers: *twenty five, roughly one thousand*

(vii) Compound evaluative expressions: *roughly small or medium, small but not very (small)*

(viii) Negative evaluative expressions: *not small, not very big*

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TE-adjectives form **antonyms:**

$$\langle\text{nominal adjective}\rangle \longleftrightarrow \langle\text{antonym}\rangle$$

young $\longleftrightarrow$ *old; ugly* $\longleftrightarrow$ *nice; stupid* $\longleftrightarrow$ *clever*

Fundamental evaluative trichotomy

⟨empty hedge⟩⟨nominal adjective⟩ —
⟨empty hedge⟩⟨middle member⟩ —
⟨empty hedge⟩⟨antonym⟩

young — medium age — old; ugly — normal — nice; stupid — medium intelligent — clever

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Evaluative linguistic predications

- Proposition: ⟨noun⟩ is $\mathscr{A}$
- Construction: ⟨noun⟩ is $\mathscr{A}$ is taken as synonymous with $\mathscr{A}$ ⟨noun⟩

Example

"*temperature is low*" synonymous with "*low temperature*"
"*this boy is very intelligent*" synonymous with "*very intelligent boy*"

more or less weak force, medium tension, extremely long bridge

short distance and pleasant walk

roughly small or medium speed

Abstract form used in applications:

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Semantics of evaluative expressions

- Context (possible world)

$$w = \langle v_L, v_S, v_R \rangle$$

Town: $\langle 3\ 000, 50\ 000, 1\ 000\ 000 \rangle$ *(Czech Republic)*
$\langle 30\ 000, 200\ 000, 10\ 000\ 000 \rangle$ *(USA)*

- Intension — a property; different truth values in various contexts; (*invariant with respect to various contexts*)
  Small town (30 000)= 0.7 (Cz)
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- Extension — fuzzy set of elements determined by an intension in a given context; (*it does change when changing the context*)

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Formal theory

- Formal theory T^{Ev} in the language of FTT, 11 special axioms
- Formalization of 5 general characteristics

Formal syntactical proofs of all properties!

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Formal syntactical proofs of all properties!

Axioms of T^{Ev}

(EV1) $(\exists z)\boldsymbol{\Delta}(\neg z \equiv z)$

(EV2) $(\bot \equiv w^{-1}\bot_w) \wedge (\dagger \equiv w^{-1}\dagger_w) \wedge (\top \equiv w^{-1}\top_w)$

(EV3) $t \sim t$

(EV4) $t \sim u \equiv u \sim t$

(EV5) $t \sim u \,\&\, u \sim z \cdot \Rightarrow t \sim z$

(EV6) $\neg(\bot \sim \dagger)$

(EV7) $\boldsymbol{\Delta}((t \Rightarrow u)\,\&\,(u \Rightarrow z)) \Rightarrow \cdot t \sim z \Rightarrow t \sim u$

(EV8) $t \equiv t' \,\&\, z \equiv z' \Rightarrow \cdot t \sim z \Rightarrow t' \sim z'$

(EV9) $(\exists u)\hat{\Upsilon}(\bot \sim u) \wedge (\exists u)\hat{\Upsilon}(\dagger \sim u) \wedge (\exists u)\hat{\Upsilon}(\top \sim u)$

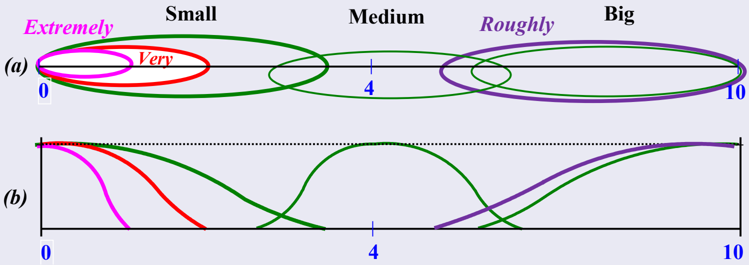
(EV10) $\mathit{NatHedge}\, \bar{\boldsymbol{\nu}} \,\&\, (\exists \boldsymbol{\nu})(\exists \boldsymbol{\nu}')(\mathit{Hedge}\, \boldsymbol{\nu} \,\&\, \mathit{Hedge}\, \boldsymbol{\nu}' \,\&\, (\boldsymbol{\nu}_1 \preceq \bar{\boldsymbol{\nu}} \wedge \bar{\boldsymbol{\nu}} \preceq \boldsymbol{\nu}_2))$

(EV11) $(\forall z)((\Upsilon\bar{\boldsymbol{\nu}}(LH\, z)) \vee (\Upsilon\bar{\boldsymbol{\nu}}(MH\, z)) \vee (\Upsilon\bar{\boldsymbol{\nu}}(RH\, z)))$

Extension of simple evaluative expressions

(A)

Extensions are classes of elements taken from nonempty, linearly ordered and bounded scales;
left bound, right bound, central point



Formal theory of evaluative expressions (context)

Context is a function $w : [0,1] \longrightarrow M$:

$$
\begin{aligned}
w(0) &= v_L \\
w(0.5) &= v_S && \text{(central point)} \\
w(1) &= v_R && \text{(right bound)}
\end{aligned}
$$

Linear ordering $\leq_w$ in each context w

Set of contexts $W = \{w \mid w : [0,1] \longrightarrow M\}$

Formal theory of evaluative expressions (horizon)

(B)

Vagueness of the meaning of natural language expressions is a consequence of indiscernibility between objects which determines three horizons on each scale (context):

Indiscernibility: fuzzy equality $\approx_w$, determined by $\sim$ on the set of truth values

Example

Standard Łukasiewicz MV-algebra of truth values:

$$[a \sim b] = \frac{0 \vee (0.5 - |a - b|)}{0.5}.$$

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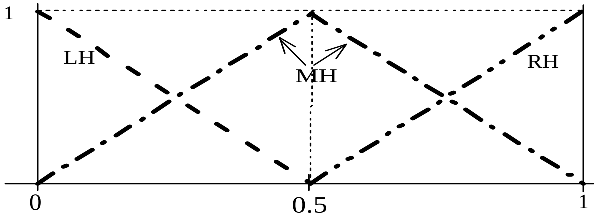
Three horizons

$$LH(a) = [0 \sim a], \qquad LH_w(x) = [v_L \approx_w x]$$
$$MH(a) = [0.5 \sim a], \qquad MH_w(x) = [v_S \approx_w x]$$
$$RH(a) = [1 \sim a], \qquad RH_w(x) = [v_R \approx_w x]$$



(C)

Horizon — analogous features as Sorites paradox

Set of natural numbers $\mathbb{N}$

Context w_N:

$$\langle v_L = w(0) = 0, v_S = w(0.5) = p, v_R = w(1) = q \rangle$$

$$\mathbb{FN}(n) = [0 \approx_{w_N} n]$$

Finite numbers do not form a heap

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(C)

Horizon — analogous features as Sorites paradox

Set of natural numbers $\mathbb{N}$
Context w_N:

$$\langle v_L = w(0) = 0, v_S = w(0.5) = p, v_R = w(1) = q\rangle$$

$$\mathbb{FN}(n) = [0 \approx_{w_N} n]$$

Finite numbers do not form a heap

Formal theory of evaluative expressions (horizon)

Theorem

(a) $\vdash (\forall w_{\epsilon o})(\mathbb{FN}\, w_{\epsilon o} 0)$

(0 stones do not form a heap in any context)

(b) $\vdash (\forall w_{\epsilon o})(\forall n)(n \in w_{\epsilon o} \,\&\, \Delta(w_{\epsilon o} 0.5 \leq n) \Rightarrow \neg \mathbb{FN}\, w_{\epsilon o} n)$

(whatever number $n \geq p$ of stones forms a heap)

(c) $\vdash (\forall w_{\epsilon o})(\exists m)(m \in w_{\epsilon o} \,\&\, 0 < m \,\&\, \hat{\Upsilon}(\mathbb{FN}\, w_{\epsilon o} m))$

(there is a borderline number of stones "partially" forming a heap)

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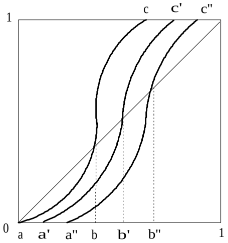
(D)

Extension of evaluative expression is a result of shifted horizon;
The shift corresponds to a linguistic hedge
(the shift is: "small for big truth values" and "big for small ones")

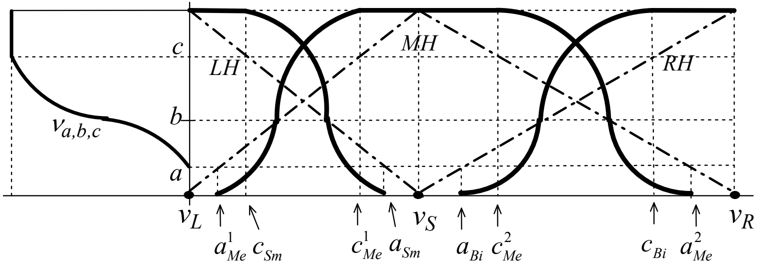
Hedges (horizon shifts): $\nu : [0, 1] \longrightarrow [0, 1]$



(E)

Each scale (context) is vaguely partitioned by the fundamental evaluative trichotomy

Special hedge $\bar{\nu}$ — empty hedge



Intensions of evaluative predications

Intension — a function from a set of contexts into a set of fuzzy sets

$$W \longrightarrow \mathscr{F}(w([0,1]))$$

Intension formally

- $\mathrm{Int}(X$ is ⟨linguistic hedge⟩ small$) := \underbrace{\lambda w\, \lambda x \cdot \boldsymbol{\nu}(LH(w^{-1}x))}_{Sm\,\boldsymbol{\nu}}$
- $\mathrm{Int}(X$ is ⟨linguistic hedge⟩medium$) := \underbrace{\lambda w\, \lambda x \cdot \boldsymbol{\nu}(MH(w^{-1}x))}_{Me\,\boldsymbol{\nu}}$
- $\mathrm{Int}(X$ is ⟨linguistic hedge⟩ big$) := \underbrace{\lambda w\, \lambda x \cdot \boldsymbol{\nu}(RH(w^{-1}x))}_{Bi\,\boldsymbol{\nu}}$

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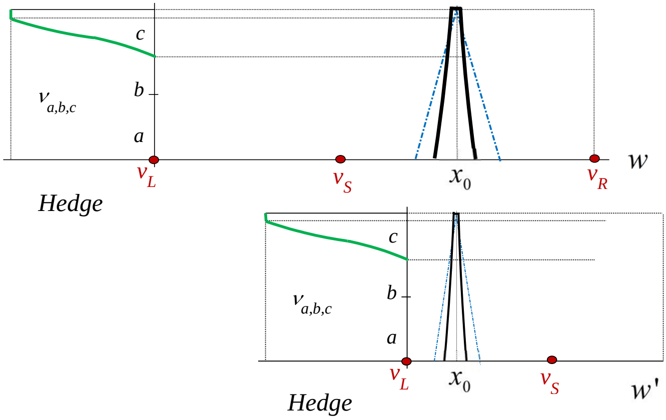
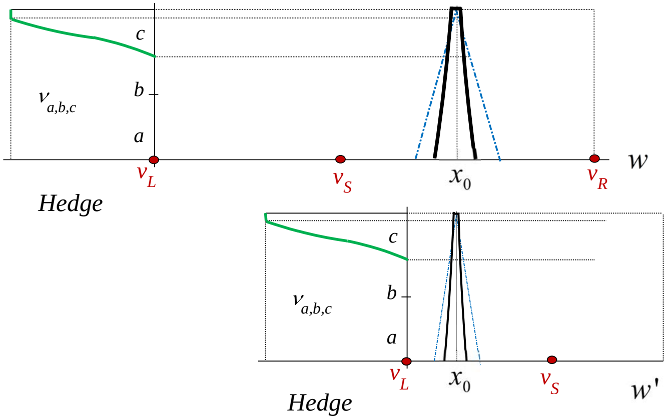
Intension of a fuzzy number "About x_0"



Theorem

The formal theory of evaluative linguistic expressions is consistent

Intension of a fuzzy number "About x_0"



Theorem

The formal theory of evaluative linguistic expressions is consistent

All, Most, Almost all, Few, Many, Some, No, etc.
Important constituent of generalized quantifiers

Main idea:

Classical quantifiers $\forall$ and $\exists$ taken over a class of elements determined using an appropriate evaluative expression.

Classical logic: *No substantiation why and how the range of quantification should be made smaller*

Intermediate quantifiers — motivation

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Important constituent of generalized quantifiers

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Classical quantifiers $\forall$ and $\exists$ taken over a class of elements determined using an appropriate evaluative expression.

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T^{IQ} — special theory of FTT

$\mu \in Form_{o(o\alpha)(o\alpha)}$, $\alpha \in \mathscr{S}$ represents a measure on fuzzy sets

Not fully general theory

Set of special types $\mathscr{S}$: difficulties with interpretation of the measure μ over sets of very large cardinalities

What does it mean "most X's" over a set of inaccessible cardinality?

Formal theory of intermediate quantifiers

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Formal theory of intermediate quantifiers

$Ev \in Form_{oo}$ — evaluative expression (intension)
$A, B, z \in Form_{o\alpha}$, $x \in Form_{\alpha}$ variables where $\alpha \in \mathscr{S}$.

Definition (Type $\langle 1, 1 \rangle$ intermediate generalized quantifier)

Interprets the sentence

"⟨Quantifier⟩ B's are A"

is one of the following formulas:

$$(Q^{\forall}_{Ev}\, x)(B, A) := (\exists z)((\mathbf{\Delta}(z \subseteq B)\, \&\, (\forall x)(z\, x \Rightarrow Ax)) \wedge Ev((\mu B)z)),$$
$$(Q^{\exists}_{Ev}\, x)(B, A) := (\exists z)((\mathbf{\Delta}(z \subseteq B)\, \&\, (\exists x)(zx \wedge Ax)) \wedge Ev((\mu B)z)).$$

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Basic definitions

$$\textbf{A:}\ \text{All } B \text{ are } A := Q^{\forall}_{Bi\boldsymbol{\Delta}}(B, A) \equiv (\forall x)(Bx \Rightarrow Ax),$$

$$\textbf{E:}\ \text{No } B \text{ are } A := Q^{\forall}_{Bi\boldsymbol{\Delta}}(B, \neg A) \equiv (\forall x)(Bx \Rightarrow \neg Ax),$$

$$\textbf{P:}\ \text{Almost all } B \text{ are } A := Q^{\forall}_{Bi\ Ex}(B, A) \equiv$$
$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\ \&\ (\forall x)(zx \Rightarrow Ax)) \wedge (Bi\ Ex)((\mu B)z)),$$

$$\textbf{B:}\ \text{Few } B \text{ are } A\ (:= \text{Almost all } B \text{ are not } A) := Q^{\forall}_{Bi\ Ex}(B, \neg A) \equiv$$
$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\ \&\ (\forall x)(zx \Rightarrow \neg Ax)) \wedge (Bi\ Ex)((\mu B)z)),$$

$$\textbf{T:}\ \text{Most } B \text{ are } A := Q^{\forall}_{Bi\ Ve}(B, A) \equiv$$
$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\ \&\ (\forall x)(zx \Rightarrow Ax)) \wedge (Bi\ Ve)((\mu B)z)),$$

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D: Most B are not $A := Q^{\forall}_{Bi\,Ve}(B, \neg A) \equiv$

$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow \neg Ax)) \wedge (Bi\ Ve)((\mu B)z)),$$

K: Many B are $A := Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(B, A) \equiv$

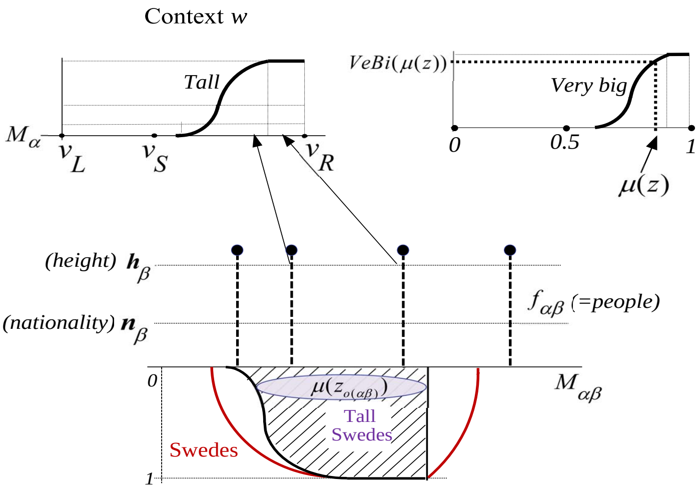
$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax)) \wedge \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z)),$$

G: Many B are not $A := Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}(B, \neg A) \equiv$

$$(\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow \neg Ax)) \wedge \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z)),$$

I: Some B are $A := Q^{\exists}_{Bi\boldsymbol{\Delta}}(B, A) \equiv (\exists x)(Bx \wedge Ax)$,

O: Some B are not $A := Q^{\exists}_{Bi\boldsymbol{\Delta}}(B, \neg A) \equiv (\exists x)(Bx \wedge \neg Ax)$.



$$Ext_w(\textit{Most Swedes are tall}) = \bigvee_z (VeBi(\mu(z)) \wedge \bigwedge_f \text{Tall-Swedes}(zf))$$

Theorem (Valid implications)

(a) $T^{IQ} \vdash \mathbf{A} \Rightarrow \mathbf{P}$, $T^{IQ} \vdash \mathbf{P} \Rightarrow \mathbf{T}$, $T^{IQ} \vdash \mathbf{T} \Rightarrow \mathbf{K}$.

(b) $T^{IQ} \vdash \mathbf{E} \Rightarrow \mathbf{B}$, $T^{IQ} \vdash \mathbf{B} \Rightarrow \mathbf{D}$, $T^{IQ} \vdash \mathbf{D} \Rightarrow \mathbf{G}$.

105 valid generalized Aristotle's syllogisms

Fuzzy/linguistic IF-THEN rule

$$\text{IF } X \text{ is } \mathscr{A} \text{ THEN } Y \text{ is } \mathscr{B}$$

$\mathscr{A}, \mathscr{B}$ — *small, very small, roughly medium, extremely big*

Conditional clause of natural language

Linguistic description $\mathscr{R}$

$$\text{IF } X \text{ is } \mathscr{A}_1 \text{ THEN } Y \text{ is } \mathscr{B}_1$$
$$\text{IF } X \text{ is } \mathscr{A}_2 \text{ THEN } Y \text{ is } \mathscr{B}_2$$
$$\dots\dots\dots\dots\dots\dots$$
$$\text{IF } X \text{ is } \mathscr{A}_m \text{ THEN } Y \text{ is } \mathscr{B}_m$$

Structured text

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Structured text

Two interpretations of linguistic description

1. Relational model

   Goal: *fuzzy approximation*

   Approximate some function known only roughly

   Reasoning method: *inspection of a fuzzy relation*
   First-order fuzzy logic: P. Hájek, I. Perfilieva, S. Gottwald, M. Daňková, L. Běhounek, V. Novák

2. Logical/linguistic model

   IF-THEN rules are conditional clauses of natural language; Linguistic description is a text which describes some situation, strategy of behavior, control of some process, etc., and which provides rules (instructions) what to do.

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Intension/extension of fuzzy/linguistic IF-THEN rules

$Ev := Sm\,\boldsymbol{\nu} \mid Me\,\boldsymbol{\nu} \mid Bi\,\boldsymbol{\nu}$

$\mathrm{Int}(X \text{ is } \mathscr{A}) = Ev^A$
$\mathrm{Int}(Y \text{ is } \mathscr{B}) = Ev^C$

Intension

$$\mathrm{Int}(\mathscr{R}) := \lambda w\,\lambda w' \cdot \lambda x\,\lambda y \cdot Ev^A wx \Rightarrow Ev^C w' y$$

Extension in a couple of contexts $w, w' \in W$

$$\mathscr{M}^{\mathscr{E}}(\mathrm{Ext}_{\langle w,w'\rangle}(\mathscr{R}))(v, v') = \mathscr{M}^{\mathscr{E}}(Ev^A w)(v) \to \mathscr{M}^{\mathscr{E}}(Ev^C w')(v')$$

$v \in w, v' \in w'$

This definition complies with the theory of conditionals

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Outline

Outline of a possible mathematical model

Semantics of basic lexical units:

- Nouns
- Adjectives
- Adverbs
- Verbs

Special types:

(i) α — subscripts of features (characteristics)

(ii) $\beta = o\gamma$ — sets of values of features

(iii) γ — all values of all features

Nouns are names of (classes of) objects

Objects

Sets (vectors) of values of features (age, height, length, weight, color, etc.)

$$\mathscr{M}(f_{\gamma\alpha}) = \{\langle j_\alpha, v_\gamma \rangle, \langle j'_\alpha, v'_\gamma \rangle, \ldots\}$$

$j_\alpha, j'_\alpha, \ldots$ are various (subscripts of) characteristics
$v_\gamma, v'_\gamma, \ldots$ are their values

Grammatemes of nouns

(i) Delimitation

1. Indefinite
2. Specifying
3. Definite

(ii) Number

1. Plural
2. Singular

The meaning of complex unit **Noun**

$N_{o(\gamma\alpha)(\tau\omega)\alpha}$ — property of objects named by the given noun

$M(\langle\textbf{noun}\rangle, \{\textbf{indefinite, plural}\})$

$$\lambda w\, \lambda t \cdot \underbrace{\lambda f_{\gamma\alpha}\, \underbrace{(\forall j_\alpha) \cdot (N_{o(\gamma\alpha)(\tau\omega)\alpha}\, w\, t\, j) f}_{o}}_{o(\gamma\alpha)}$$

Example

Swede := $\lambda w\, \lambda t \cdot \lambda f_{\gamma\alpha}\, (\forall j_\alpha) \cdot (\mathrm{Sw}_{o(\gamma\alpha)\tau\omega\alpha}\, w\, t\, j) f$

Extension of Swedes in a given possible world w and time t:
fuzzy set of people f with the nationality being Swede
$\mathrm{Ext}(\mathrm{Swede})(wt) := \lambda f_{\gamma\alpha}\, (\forall j_\alpha)(\mathrm{Sw}\, wtj) f$

The meaning of complex unit **Noun**

$M(\langle \textbf{noun} \rangle, \{\textbf{definite, singular}\})$

Concrete object

$$\lambda w \, \lambda t \, \mathbf{f}_{\gamma\alpha}$$

characterized by truth degree of

$$\underbrace{(\forall j_\alpha) \cdot (N_{o(\gamma\alpha)\tau\omega} \, wt \, j)\mathbf{f}}_{o}$$

for the given w, t

"The Swede we spoke about"

Meaning of the complex unit **Noun**

$M(\langle\textbf{noun}\rangle, \{\textbf{specifying, singular}\})$

$$\lambda w\,\lambda t \cdot \underbrace{\iota_{(\gamma\alpha)(o(\gamma\alpha))}\, \underbrace{\lambda f_{\gamma\alpha}\, \underbrace{(\forall j_\alpha)\cdot(N_{o(\gamma\alpha)\tau\omega\alpha}\, wtj)f}_{o}}_{o(\gamma\alpha)}}_{\text{element } \hat{f}_{\gamma\alpha} \text{ from the kernel}}$$

"A Swede entered the room"

Adjectives

Grammateme of degree:

1. Positive
2. Comparative
3. Superlative

$A_{o\gamma(\tau\omega)}$ — original adjectives are names of features of objects

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$M(\langle\textbf{adjective}\rangle, \{\textbf{positive}\})$

$$\lambda w\, \lambda t \cdot \underbrace{\lambda v_\gamma\, \underbrace{(A_{o\gamma(\tau\omega)}\, w\, t)v}_{o}}_{o\gamma}$$

Example

$\mathbf{j}_H$ — feature *Height*

$$\text{Tall Swedes} := \lambda w\, \lambda t\, \lambda f_{\gamma\alpha} \cdot (\text{Swede}\ wt)f\ \&(Bi\ \ wt)(f\ \mathbf{j}_H)$$

Adjectives

$M(\langle\textbf{adjective}\rangle, \{\textbf{positive}\})$

$$\lambda w\, \lambda t \cdot \underbrace{\lambda v_\gamma\, \underbrace{(A_{o\gamma(\tau\omega)}\, w\, t)v}_{o}}_{o\gamma}$$

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$\mathbf{j}_H$ — feature *Height*

$$\text{Tall Swedes} := \lambda w\, \lambda t\, \lambda f_{\gamma\alpha} \cdot (\text{Swede}\; wt)f \;\&\; (Bi\;\; wt)(f\, \mathbf{j}_H)$$

Adjectives

A feature induces ordering: $(M_\gamma, \geq)$
Comparative of adjective: name of this ordering $A^{\geq}_{(o\gamma)\gamma(\tau\omega)}$

$M(\langle$adjective$\rangle, \{$comparative$\})$

$$\lambda w\, \lambda t \cdot \underbrace{\lambda v_\gamma \lambda v'_\gamma\, \underbrace{(A^{\geq}_{(o\gamma)\gamma(\tau\omega)}\, w\, t) v\, v'}_{o}}_{(o\gamma)\gamma}$$

Example

John ($\mathrm{John}_{\gamma\alpha}$) is taller than Marry ($\mathrm{Marry}_{\gamma\alpha}$)

$$\underbrace{(\mathrm{Tall}^{\geq}\, wt)(\mathrm{John}\, \mathbf{j}_H)(\mathrm{Marry}\, \mathbf{j}_H)}_{o}$$

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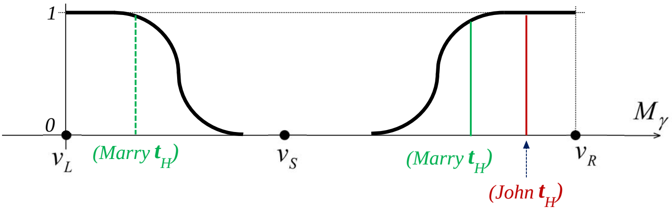
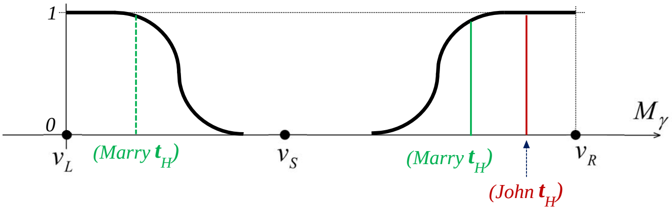
John is taller than Marry



John is tall $\not\Rightarrow$ Marry is tall!
Marry is tall $\Rightarrow$ John is tall

No explicit relation between *tall* and *taller*!

John is taller than Marry



John is tall $\not\Rightarrow$ Marry is tall!
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Adjectives

$M(\langle\textbf{adjective}\rangle, \{\textbf{superlative}\})$

Concrete object

$$\lambda w\, \lambda t\, \mathbf{v}_\gamma$$

characterized by truth degree of

$$\boldsymbol{\Delta}(\mathbf{v} \equiv \iota_{\gamma(o\gamma)}\lambda v_\gamma\, (A_{o\gamma(\tau\omega)}\, w\, t)v) \wedge (\forall v)((A^{\geq}_{(o\gamma)\gamma(\tau\omega)}\, w\, t)\mathbf{v}\, v)$$

The tallest (smallest) boy in the classroom

Adverbs

Non-homogenous class

They express more closely circumstances of various phenomena, modify properties, their meaning varies dependently on their relation to other syntagms; quite often they are derived on the surface level from adjectives

Linguistic hedges (above)

Grammatemes of verbs

(i) sentential modality

1. Indicative
2. Interrogative
3. Imperative

(ii) Aspect

1. Processual
2. Complex
3. Resultative

(iii) Extension

1. Immediate
2. Extended (atemporal)

(iv) Tense

1. Simultaneous
2. Anterior
3. Posterior

Grammatemes of verbs

(i) Iterativeness
1. Iterative
2. Noniterative

(ii) Predicate modality
1. Indicative
2. Necessity
3. Desirability
4. Possibility

(iii) Assertive modality
1. Positive
2. Negative

Other characteristics of verbs

- Semantic variations (locative, directional, before, after, etc.)
- Valency frames (participants and modifications)

Require proper model of time

Versatile Event Logic (Bennett, Galton)

- Explicit reference to time points and a temporal ordering relation
- Explicit reference to temporal intervals and the temporal relationships between them
- Use of propositional tenses to convey temporal relationsships among facts

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Interpretation

(Sets of) fuzzy sets of fuzzy relations (?)

How to distinguish mathematically

$$\textit{John goes to school: } Go(John, school)$$

from

$$\textit{John writes homework: } Write(John, homework)$$

VEL: Set S of world states
set H of histories; each $h \in H$

$$h : T \longrightarrow S$$

Event — sequence of states;
branching time — history tree

Example (L. A. Zadeh)

Usually, it takes Robert about an hour to get home from work

Outline

Conclusions

- MFL is a very general and powerful tool for many purposes
- Higher-order fuzzy logic can be efficiently used for modeling of linguistic semantics
- There is no unified formal theory of linguistic semantics (no theory T with one set of axioms)
- Reasoning based on linguistic semantics requires non-monotonicity (FLb)

What do we want from our theory?

- Formal tool enabling logical analysis of natural language
- Tool enabling reasoning based on knowledge expressed in natural language
- Working model of linguistic semantics for practical purposes

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- Higher-order fuzzy logic can be efficiently used for modeling of linguistic semantics
- There is no unified formal theory of linguistic semantics (no theory T with one set of axioms)
- Reasoning based on linguistic semantics requires non-monotonicity (FLb)

What do we want from our theory?

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