

# Priestley Duality and Nelson Lattices

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# Historical remarks

- [M. Stone 1936]  
Duality of Boolean algebras and Stone spaces  
(compact totally disconnected spaces)
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“Less satisfactory” duality for distributive lattices
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Priestley duality for **N3**-Lattices (models of explosive Nelson’s logic **N3**) based on interpolation property [Monteiro 1963]

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# Priestley Spaces

- $\mathcal{X} = (X, \leq, \tau)$  is an **ordered topological space** if  $\leq$  is a partial order on  $X$  and  $\tau$  a topology on  $X$ .
- $\mathcal{X}$  is a **totally order-disconnected topological space** if for any  $x, y \in X$  with  $x \not\leq y$ , there is a clopen cone  $U$  such that  $x \in U$  and  $y \notin U$ .
- **Priestley space** is a **compact** totally order-disconnected topological space.
- $\mathcal{T}$  is the category of Priestley spaces and order preserving continuous functions.

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# Distributive lattices $\rightarrow$ Priestley spaces

- $\mathcal{D}$  is the category of bounded distributive lattices and their homomorphisms.
- For  $\mathcal{A} \in \text{Ob}(\mathcal{D})$  define

$$X(\mathcal{A}) = (X, \subseteq, \tau), \text{ where}$$

- 1  $X$  is the set of prime filters on  $\mathcal{A}$ ;
- 2  $\subseteq$  is the set inclusion;
- 3  $\tau$  is given by the **subbase**:

$$\sigma_{\mathcal{A}}(a) := \{P \in X \mid a \in P\} \text{ and } X \setminus \sigma_{\mathcal{A}}(a), \text{ where } a \in \mathcal{A}$$

- For  $f : \mathcal{A} \rightarrow \mathcal{B}$ , define  $X(f) : X(\mathcal{B}) \rightarrow X(\mathcal{A})$  by:

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# Priestley Duality

- $X : \mathcal{D} \rightarrow \mathcal{T}$  and  $D : \mathcal{T} \rightarrow \mathcal{D}$  are contravariant functors.
- For  $\mathcal{X} \in \text{Ob}(\mathcal{T})$ ,  $\varepsilon_{\mathcal{X}} : \mathcal{X} \rightarrow XD(\mathcal{X})$  is a  $\mathcal{T}$ -isomorphism.

$$\varepsilon_{\mathcal{X}}(x) := \{U \in D(\mathcal{X}) \mid x \in U\}$$

- For  $\mathcal{A} \in \text{Ob}(\mathcal{D})$ ,  $\sigma_{\mathcal{A}} : \mathcal{A} \rightarrow DX(\mathcal{A})$  is a lattice isomorphism.

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# Priestley duality

- Moreover  $\sigma : \mathbf{1}_{\mathcal{D}} \cong \mathbf{D} \circ \mathbf{X}$  and  $\varepsilon : \mathbf{1}_{\mathcal{T}} \cong \mathbf{X} \circ \mathbf{D}$  are **natural isomorphisms**, i.e., for any  $\mathcal{D}$ -morphism  $f : \mathcal{A} \rightarrow \mathcal{B}$  and  $\mathcal{T}$ -morphism  $g : \mathcal{X} \rightarrow \mathcal{Y}$ , the diagrams below are commutative.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\sigma_{\mathcal{A}}} & \mathbf{DX}(\mathcal{A}) \\ \downarrow f & & \downarrow \mathbf{DX}(f) \\ \mathcal{B} & \xrightarrow{\sigma_{\mathcal{B}}} & \mathbf{DX}(\mathcal{B}) \end{array}$$

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\varepsilon_{\mathcal{X}}} & \mathbf{XD}(\mathcal{X}) \\ \downarrow g & & \downarrow \mathbf{XD}(f) \\ \mathcal{Y} & \xrightarrow{\varepsilon_{\mathcal{Y}}} & \mathbf{XD}(\mathcal{Y}) \end{array}$$

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# Duality for Heyting algebras (folklore)

- **Heyting space** is a Priestley space  $\mathcal{X}$  s. t.  $U \downarrow$  is open for any  $U$  open in  $\mathcal{X}$ .
- For a Heyting space  $\mathcal{X}$ ,  $D_H(\mathcal{X})$  is the lattice  $D(\mathcal{X})$  with the operation  $\supset$ :

$$U \supset V := X \setminus (U \setminus V) \downarrow \text{ for any } U, V \in D(\mathcal{X}).$$

$D_H(\mathcal{X})$  is a Heyting algebra.

- $\mathcal{T}$ -morphism  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is a **Heyting function** if
$$f^{-1}(U \downarrow) = (f^{-1}(U)) \downarrow \text{ for any } U \text{ open in } \mathcal{Y}.$$
- Categories  $\mathcal{H}^*$  of Heyting spaces and functions and  $\mathcal{H}$  of Heyting algebras and their homomorphisms are dually equivalent via  $X_H$  and  $D_H$ .

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# Priestley duality for De Morgan algebras

## [Cornish & Fowler 77]

- A De Morgan space is a pair  $(\mathcal{X}, g)$ , where
  - $\mathcal{X}$  is a Priestley space;
  - $g : \mathcal{X} \rightarrow \mathcal{X}$  is an antimonotonic homeomorphism s.t.  $g^2 = id_{\mathcal{X}}$ .
- $f : (\mathcal{X}, g) \rightarrow (\mathcal{X}', g')$  is a De Morgan function if  $fg = g'f$ .
- For a De Morgan space  $(\mathcal{X}, g)$ , put:

$$M(\mathcal{X}, g) := \langle D(\mathcal{X}), \cap, \cup, \sim, \emptyset, X \rangle, \text{ where}$$

$$\sim U := X \setminus g(U).$$

- For a De Morgan algebra  $\mathcal{A}$ , put  $S(\mathcal{A}) := (X(\mathcal{A}), g)$ , where

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# Axiomatics of **N4**

- Language  $\langle \vee, \wedge, \rightarrow, \sim \rangle$ , where  $\sim$  is a **strong negation**.

- Axioms of **positive intuitionistic logic**:

$$\mathbf{P1.} \quad \alpha \rightarrow (\beta \rightarrow \alpha)$$

$$\mathbf{P2.} \quad (\alpha \wedge \beta) \rightarrow \alpha$$

$$\mathbf{P3.} \quad (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$$

$$\mathbf{P4.} \quad (\alpha \wedge \beta) \rightarrow \beta$$

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$$\mathbf{P8.} \quad \beta \rightarrow (\alpha \vee \beta)$$

- **Strong negation axioms**:

$$\mathbf{N1.} \quad \sim (\alpha \rightarrow \beta) \leftrightarrow \alpha \wedge \sim \beta$$

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$$\mathbf{N3.} \quad \sim \sim \alpha \leftrightarrow \alpha$$

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- Inference rule:

$$MP \quad \frac{\alpha, \alpha \rightarrow \beta}{\beta}$$

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# Logics **N3** and **N4**<sup>⊥</sup>

- **N3** := **N4** + { $\sim p \rightarrow (p \rightarrow q)$ }
- **N4**<sup>⊥</sup> in the language  $\langle \vee, \wedge, \rightarrow, \sim, \perp \rangle$  with additional axioms

$$\perp \rightarrow p \text{ and } p \rightarrow \sim \perp$$

- Put  $\perp := \sim (p_0 \rightarrow p_0)$ . Then **N3**  $\vdash \perp \rightarrow p, p \rightarrow \sim \perp$

- Constructive Negation Property

$$\mathbf{N4} \vdash \sim (\varphi \wedge \psi) \Rightarrow \mathbf{N4} \vdash \sim \varphi \text{ or } \mathbf{N4} \vdash \sim \psi$$

- Weak Replacement for Nelson logic

$$\frac{\varphi \leftrightarrow \psi \quad \sim \varphi \rightarrow \sim \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)}$$

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$$\perp \rightarrow p \text{ and } p \rightarrow \sim \perp$$

- Put  $\perp := \sim (p_0 \rightarrow p_0)$ . Then **N3**  $\vdash \perp \rightarrow p, p \rightarrow \sim \perp$
- Constructive Negation Property

$$\mathbf{N4} \vdash \sim (\varphi \wedge \psi) \Rightarrow \mathbf{N4} \vdash \sim \varphi \text{ or } \mathbf{N4} \vdash \sim \psi$$

- Weak Replacement for Nelson logic

$$\frac{\varphi \leftrightarrow \psi \quad \sim \varphi \rightarrow \sim \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)}$$

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# $\mathbf{N4}^\perp$ -lattices [Odintsov05]

$\mathcal{A} = \langle \mathbf{A}, \vee, \wedge, \rightarrow, \sim, \perp, 1 \rangle$  is an  $\mathbf{N4}^\perp$ -lattice if:

- 1  $\langle \mathbf{A}, \vee, \wedge, \sim, \perp, 1 \rangle$  is a De Morgan algebra;
- 2  $\preceq$ , where  $a \preceq b$  denotes  $(a \rightarrow b) \rightarrow (a \rightarrow b) = a \rightarrow b$ , is a preordering on  $\mathcal{A}$ ;
- 3  $\approx := \preceq \cap \preceq^{-1}$  is a congruence wrt  $\vee, \wedge, \rightarrow$  and  $\mathcal{A}_\approx := \langle \mathbf{A}, \vee, \wedge, \rightarrow, \perp, 1 \rangle / \approx$  is a Heyting algebra;
- 4  $\sim (a \rightarrow b) \approx a \wedge \sim b$ ;
- 5  $a \leq b$  if and only if  $a \preceq b$  and  $\sim b \preceq \sim a$ .

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## N3-lattices [Rasiowa58]

- 1 An **N3**-lattice  $\mathcal{A}$  is an **N4**<sup>⊥</sup>-lattice s.t.  $a \wedge \sim a \preceq b$  for  $a, b \in \mathcal{A}$
- 2 in which case  $a \preceq b \Leftrightarrow a \rightarrow b = 1$
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- 4 weak implication via relative pseudo complement in **N3**-lattices  
 $a \rightarrow b = a \supset (\sim a \vee b)$ , where  $a \supset b := \sup\{x \mid a \wedge x \leq b\}$
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# Special filters [Rasiowa58] on $\mathbf{N4}^\perp$ -lattices

- Let  $\mathcal{A}$  be an  $\mathbf{N4}^\perp$ -lattice and  $\emptyset \neq F \subseteq A$ .
- $F$  is a **special filter of the first kind** (*sffk*) on  $\mathcal{A}$  if:
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# De Morgan space of an $\mathbf{N4}^\perp$ -lattice $\mathcal{A}$

- 1 Since  $\mathcal{A}$  is a De Morgan algebra,  $(X(\mathcal{A}), g)$ , where  $g(P) = \mathcal{A} \setminus \tilde{P}$ ,  $\tilde{P} = \{\sim a \mid a \in P\}$ , is a De Morgan space.

- 2 Put

$$X^1(\mathcal{A}) = \{P \in X(\mathcal{A}) \mid P \text{ is a sffk}\}$$

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$$X^+(\mathcal{A}) = \{P \in X(\mathcal{A}) \mid P \subseteq g(P)\}$$

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- 3  $g(X^1(\mathcal{A})) = X^2(\mathcal{A})$  and  $g(X^+(\mathcal{A})) = X^-(\mathcal{A})$

- 4  $\mathcal{A}$  is an  $\mathbf{N3}$ -lattice iff

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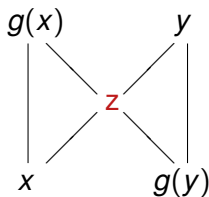
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# $N3$ -spaces [Cignoli 86, Sendlewski 90]

$\mathcal{X} = (X, \leq, \tau, g)$ , where  $X$  is a set,  $\leq$  a partial order on  $X$ ,  $g : X \rightarrow X$ , and  $\tau$  is a topology on  $X$ , is an  $N4$ -space if:

- 1  $(X, \leq, \tau, g)$  is a De Morgan space;
- 2  $(X^+, \leq|_{X^+}, \tau|_{X^+})$  is a Heyting space, where  $X^+ := \{x \mid x \leq g(x)\}$ ;
- 3 (**Monteiro Interpolation Property**) for any  $x \in X^+$  and  $y \in X^-$ , if  $x \leq y$ , then there exists  $z \in X$  s.t.  $x, g(y) \leq z \leq g(x), y$



# $N4$ -spaces

Consider a tuple  $\mathcal{X} = (X, X^1, \leq, \tau, g)$ , where  $X$  is a set,  $X^1 \subseteq X$ ,  $\leq$  is a partial order on  $X$ ,  $g : X \rightarrow X$ , and  $\tau$  is a topology on  $X$ . Put

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# Duality

- 1 Let  $\mathcal{A}$  be an  $\mathbf{N4}^\perp$ -lattice. Then

$$\mathcal{O}(\mathcal{A}) := (X(\mathcal{A}), X^1(\mathcal{A}), \subseteq, \tau_{\mathcal{A}}, g_{\mathcal{A}}),$$

where

- $X(\mathcal{A})$  are all prime filters on  $\mathcal{A}$
- $X^1(\mathcal{A})$  are all *sffk* in  $X(\mathcal{A})$
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- 2 Let  $\mathcal{X} = (X, X^1, \leq, \tau, g)$  be an  $N4$ -space. Then  $\mathbf{N}(\mathcal{X}) = \langle \mathbf{D}(\mathcal{X}), \cup, \cap, \rightarrow, \sim, \emptyset, X \rangle$ , where

$$\sim U := X \setminus g(U),$$

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# N4-function

$f : (X, X^1, \leq, \tau, g) \rightarrow (Y, Y^1, \leq', \tau', g')$  is an **N4-function** if:

- 1  $f : (X, \leq, \tau, g) \rightarrow (Y, \leq', \tau', g')$  is a De Morgan function, i.e., an order preserving continuous mapping s. t.  $fg = g'f$ ;
- 2  $f(X^1) \subseteq Y^1$ ;
- 3  $f \upharpoonright_{X^1}$  is a Heyting function.