

Complete ideal completions of residuated lattices and completeness of substructural predicate logics

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I. Semantical study of QSLs

We will discuss completions of algebras in relation to algebraic completeness of **substructural predicate logics** (QSLs). For the simplicity's sake, we will consider only commutative case.

- 1 Algebraic semantics for QSLs
- 2 Regular completions
- 3 Infinite distributivity
- 4 Complete ideal completions
- 5 Logical consequences
- 6 Kripke-type semantics

I.1 Residuated lattices

An algebra $\mathbf{A} = \langle A, \vee, \wedge, \cdot, \rightarrow, 1 \rangle$ is a *commutative residuated lattice* (CRL), iff it satisfies the following:

- $\langle A, \vee, \wedge \rangle$ is a lattice,
- $\langle A, \cdot, 1 \rangle$ is a commutative monoid,
- for all $x, y, z \in A$, $x \cdot y \leq z$ iff $y \leq x \rightarrow z$.

We say that (\cdot, \rightarrow) forms a *residuated pair*.

An **FL_e-algebra** is a CRL with a fixed element 0. Using 0, *negation* $-$ is defined by $-x = x \rightarrow 0$.

1.2 Algebraic semantics

Our *algebraic frames* for QSLs are structures of the form $\langle \mathbf{A}, V \rangle$ such that \mathbf{A} is a **complete** \mathbf{FL}_e -algebra and V a non-empty set, called the individual domain. Algebras are said to be complete when the join $\bigwedge X$ and the meet $\bigvee X$ exist always for each subset X .

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In each valuation f , quantifiers are interpreted by **infinite meets** and **infinite joins**, resp., as follows.

- $f(\forall x\varphi(x)) = \bigwedge \{f(\varphi(w)) : w \in V\}$
- $f(\exists x\varphi(x)) = \bigvee \{f(\varphi(w)) : w \in V\}$

A QSL \mathbf{L} is (*algebraically*) *complete* with respect to a class \mathcal{K} of algebraic frames when for each formula φ , $\varphi \in \mathbf{L}$ iff :

$\langle \mathbf{A}, V \rangle \models f(\varphi) \geq 1$ for every $\langle \mathbf{A}, V \rangle \in \mathcal{K}$ and every valuation f .

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How to show the algebraic completeness of a logic \mathbf{L} ?

How to find such complete algebras for \mathbf{L} ?

II. Regular completions

Basic idea:

- 1 Take the Lindenbaum algebra \mathbf{A} of \mathbf{L} , which is not a complete algebra.
- 2 Get a complete algebra \mathbf{C} by *completing* \mathbf{A} .

II. Regular completions

Basic idea:

- 1 Take the Lindenbaum algebra \mathbf{A} of \mathbf{L} , which is not a complete algebra.
- 2 Get a complete algebra \mathbf{C} by *completing* \mathbf{A} .

A *completion* of a given CRL (\mathbf{FL}_e -algebra) \mathbf{A} is a pair (\mathbf{C}, h) of a complete CRL (\mathbf{FL}_e -algebra) \mathbf{C} and an embedding h from \mathbf{A} to \mathbf{C} .

Often we omit h and say simply that \mathbf{C} is a completion of an algebra \mathbf{A} , whenever h is clear from the context.

II.1 Regular completions

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One of general schemes of getting completions is to use *nuclei*.

II.2 Nuclei

Let \mathbf{P} be a (commutative) p.o. monoid. An operation E on $\wp(\mathbf{P})$ is a *nucleus* iff it is a closure operator satisfying that $E(X) \cdot E(Y) \subseteq E(X \cdot Y)$. Here, $U \cdot V = \{u \cdot v : u \in U \text{ and } v \in V\}$.

Let $\wp(P)_E$ be the set of all *E -closed* subsets X of P , i.e. subsets X such that $E(X) = X$. For all $X, Y \in \wp(P)$, define

- $X \vee_E Y = E(X \cup Y)$ and $X \wedge Y = X \cap Y$
- $X \circ_E Y = E(X \cdot Y)$
- $X \Rightarrow Y = \{z : z \cdot x \in Y \text{ for all } x \in X\}$.

Let $\mathbf{P}^E = \langle \wp(P)_E, \vee_E, \wedge, \circ_E, \Rightarrow \rangle$. Then, \mathbf{P}^E forms a *complete* CRL. Thus, each nucleus over a p.o. monoid gives us a complete CRL. $E(\{1\})$ and $E(\{0\})$ give the unit and the zero element, respectively.

For a given CRL \mathbf{A} , consider its p.o.monoid-reduct \mathbf{P}_A . Sometimes, $(\mathbf{P}_A)^E$ is denoted by \mathbf{A}^E by identifying \mathbf{P}_A with \mathbf{A} .

11.3 MacNeille completions

For each subset X of a CRL \mathbf{A} , $U(X)$ ($L(X)$) denotes the set of all upper bounds (lower bounds, resp.) of X . Define $M(X) = L(U(X))$.

Then, M is a nucleus, and hence \mathbf{A}^M is a complete CRL. Moreover, the mapping h defined by $h(a) = [a] = \{x : x \leq a\}$, called the *principal mapping*, is a regular (RL-) embedding from \mathbf{A} to \mathbf{A}^M .

The complete CRL \mathbf{A}^M (with the mapping h) is called the *MacNeille completion* of a CRL \mathbf{A} .

Using MacNeille completions, we can show the algebraic completeness of many substructural predicate logics, which are *minimum predicate extensions* of basic substructural propositional logics.

Various basic systems of QSLs, including \mathbf{QInt} , \mathbf{QFL}_{ew} , \mathbf{QFL}_e , ..., are algebraically complete. ♣ HO, 1993.

III.1 Distributivity

The distributive law holds in many important substructural logics, e.g. relevant logics, many-valued logics and fuzzy logics. On the other hand, it is well-known that the distributive law is not always preserved under MacNeille completions.

III.1 Distributivity

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Then, how to show the algebraic completeness of QSLs with the distributive law?

A case analysis:

Consider the case for intuitionistic predicate logic **QInt**. MacNeille completions of Heyting algebras are always Heyting algebras, and thus are distributive. Therefore, we get the algebraic completeness of **QInt**.

Then, why MacNeille completions can preserve always the distributivity in this case?

In general, the following distributivity of the fusion over join holds in every CRL, using the law of residuation,

$$\bigvee_i a_i \cdot b = \bigvee_i (a_i \cdot b).$$

Here, the equation means that if $\bigvee_i a_i$ exists then $\bigvee_i (a_i \cdot b)$ exists also and the left hand side is equal to the right hand side.

In any Heyting algebra, \wedge and \rightarrow forms a residuated pair. Thus, a strong form of the usual distributive law holds in it.

III.2 Infinite distributivity

The **join infinite distributivity** (JID) (or (\wedge, \vee) -Dis) and the **meet infinite distributivity** (MID) (or (\vee, \wedge) -Dis) are defined as follows:

- (JID) : $\bigvee_i a_i \wedge b = \bigvee_i (a_i \wedge b)$
- (MID) : $\bigwedge_i a_i \vee b = \bigwedge_i (a_i \vee b)$

Here, the (JID) means that if $\bigvee_i a_i$ exists then $\bigvee_i (a_i \wedge b)$ exists also and the equality holds, and similarly for the (MID).

While the (JID) holds always in Heyting algebras, the (MID) does not.

In fact, the (JID) holds in any \mathbf{FL}_e -algebra \mathbf{A} such that :

- the (div) : $a \wedge b = a \cdot (a \rightarrow b)$ holds in \mathbf{A} , or
- all of the (MID), the prelinearity $(a \rightarrow b) \vee (b \rightarrow a) \geq 1$ and $a \wedge b \geq a \cdot (a \rightarrow b)$ hold in \mathbf{A} . (This is essentially due to P. Cintula).

When the (JID) holds in a *complete* CRL, a new implication \rightsquigarrow can be introduced by

$$a \rightsquigarrow b = \bigvee \{c : a \wedge c \leq b\}.$$

Obviously, \wedge and \rightsquigarrow forms a residuated pair.

These distributive laws (JID) and (MID) correspond to the following well-known axiom schemes. Here, the variable x does not occur in ψ as a free variable.

- (\wedge, \exists) : $(\exists x \varphi(x) \wedge \psi) \rightarrow \exists x (\varphi(x) \wedge \psi)$.
- (\vee, \forall) : $(\forall x (\varphi(x) \vee \psi) \rightarrow (\forall x \varphi(x) \vee \psi)$.

IV. Complete ideal completions

In the rest of our talk, we will focus our attention on another type of regular completions, called **complete ideal completions**, which preserve infinite distributivity (JID).

First, we introduce **ideal completions**. Though they are not regular, their close relations to complete ideal completions are important.

IV.1 Ideal completions

A nonempty subset X of a CRL \mathbf{A} is an *ideal* of \mathbf{A} iff

- X is downward closed,
- If $a, b \in X$ then $a \vee b \in X$.

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For each non-empty subset Y of \mathbf{A} , let $J(Y)$ be the smallest ideal including Y . Clearly, J is a nucleus, and therefore \mathbf{A}^J is a complete CRL. Moreover, the principal mapping h is an embedding from \mathbf{A} to \mathbf{A}^J .

We call \mathbf{A}^J , the *ideal completion* of \mathbf{A} .

The mapping h in the ideal completion preserves existing infinite meets, but is not always regular. On the other hand, the distributivity is preserved under ideal completion.

If the distributive law holds in a CRL then the join infinite distributivity (JID) holds in its ideal completion. ♣ See e.g. G. Birkhoff.

IV.2 Complete ideal completions

A nonempty subset X of a CRL \mathbf{A} is an *complete ideal* of \mathbf{A} iff

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- If $a_j \in X$ for each J and moreover if $\bigvee_j a_j$ exists, then $\bigvee_j a_j \in X$

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For each non-empty subset Y of \mathbf{A} , let $K(Y)$ be the smallest complete ideal including Y . We can show that K is a nucleus, and therefore \mathbf{A}^K is a complete CRL.

In this case, the principal mapping h is a *regular* embedding from \mathbf{A} to \mathbf{A}^K .

We call \mathbf{A}^K (with the principal mapping), the *complete ideal completion* (or, Crawley completion) of \mathbf{A} .

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We call \mathbf{A}^K (with the principal mapping), the *complete ideal completion* (or, Crawley completion) of \mathbf{A} .

♣ P. Crawley, "Regular embeddings which preserve lattice structure", Proc. Amer. Math. Soc. 13, 1962.

♣ W.H. Cornish, "Crawley's completion of a conditionally upper continuous lattice", Pacific Journal of Math., 51, 1974.

IV.3 Explicit representation

For each non-empty subset Y of \mathbf{A} , let $K_0(Y) =$

$$\{z : z \leq \bigvee_i y_i \text{ for existing } \bigvee_i y_i \text{ such that } y_i \in Y \downarrow \text{ for each } i\}.$$

Then,

- In general, $K_0(Y) \subseteq K(Y)$,
- the (JID) implies $K_0(Y) = K(Y)$.

IV.4 Preservation of (JID)

Theorem

For any CRL \mathbf{A} , the (JID) holds in $\mathbf{A} \Leftrightarrow$ the (JID) holds in the complete ideal completion \mathbf{A}^K .

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Theorem

For any CRL \mathbf{A} , the (JID) holds in the MacNeille completion $\mathbf{A}^M \Rightarrow$ the (JID) holds in \mathbf{A} .

IV.5 MacNeille vs complete ideal completions

What are relations between these two regular completions?

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It is easy to see that every **normal ideal**, i.e. every M -closed subset, is a complete ideal. The converse holds when the (JID) holds in the MacNeille completion \mathbf{A}^M (cf. Cornish 1974 for lattices). In fact,

Theorem

For a CRL \mathbf{A} , the following are equivalent:

- *The (JID) holds in the MacNeille completion \mathbf{A}^M .*
- *The (JID) holds in \mathbf{A} , and every complete ideal of \mathbf{A} is normal (i.e. $\mathbf{A}^M = \mathbf{A}^K$).*

As noted before, a Heyting implication can be preserved under MacNeille completions. Thus, the following well-known result is obtained as a corollary.

Corollary

If \mathbf{A} is a Heyting algebra, then $\mathbf{A}^M = \mathbf{A}^K$.

In fact, in this case \mathbf{A}^M is also a Heyting algebra and hence the (JID) holds in it.

V. Logical consequences

Preservation of equations

- Preservation of equations under ideal completions has been discussed in G. Birkhoff 1948 for lattice identities. See also A. García-Cerdaña 2008, and R. Goldblatt 2009 in the setting of substructural propositional logics.

For a given CRL \mathbf{A} , we compare the ideal completion \mathbf{A}^J with the complete ideal completion \mathbf{A}^K and discuss what kind of equations are preserved.

V.1 Preservation results

For all J -closed X and Y , the following holds.

- $K(X) \vee_K K(Y) = K(X \vee_J Y)$ (in fact this holds also for infinite joins)
- $K(X) \wedge_K K(Y) = K(X \wedge_J Y)$ (if we assume the (JID))
- $K(X) \circ_K K(Y) = K(X \circ_J Y)$
- $K(X) \Rightarrow_K K(Y) = K(X) \Rightarrow_J K(Y) = K(X \Rightarrow_J K(Y))$.

It means that a mapping f from $\wp(A)_J$ to $\wp(A)_K$ defined by $f(X) = K(X)$ is an *upper regular, lattice-ordered monoid, onto homomorphism* if the (JID) holds in \mathbf{A} .

Instead of stating here a bit complicated preservation theorem, we show how it works for a given example. Let \mathbf{A} be an arbitrary \mathbf{FL}_e -algebra.

- Clearly, the formula $\alpha \rightarrow (\beta \rightarrow \alpha \cdot \beta)$ is valid in \mathbf{A} .
- This formula is shown to be valid also in \mathbf{A}^J .
- Therefore, for all J -closed X and Y , $X \subseteq (Y \Rightarrow_J X \circ_J Y)$.
- Then, $K(X) \subseteq K(Y \Rightarrow_J X \circ_J Y) \subseteq K(Y \Rightarrow_J K(X \circ_J Y)) = K(Y) \Rightarrow_K (K(X) \circ_K K(Y))$.
- This means that the above formula is valid in \mathbf{A}^K .

V.2 Further completeness results

Algebraic completeness of QSLs with the distributive law and the axiom (\wedge, \exists) can be shown by using complete ideal completions, where (\wedge, \exists) is

$$\exists x \varphi(x) \wedge \psi \rightarrow \exists x (\varphi(x) \wedge \psi).$$

V.2 Further completeness results

Algebraic completeness of QSLs with the distributive law and the axiom (\wedge, \exists) can be shown by using complete ideal completions, where (\wedge, \exists) is

$$\exists x \varphi(x) \wedge \psi \rightarrow \exists x (\varphi(x) \wedge \psi).$$

- algebraic completeness of **QInt**. ♣ see e.g. a proof in the book by Troelstra-van Dalen.
- a direct proof of algebraic completeness of relevant predicate logic **QDFL_e** with (\wedge, \exists) . ♣ cf. HO 1999.
- in general, algebraic completeness of minimum predicate extensions of *distributive* substructural propositional logics with *the axiom* (\wedge, \exists) , by using our preservation result.

V.3 Conservativity of Heyting implication

By a **Heyting implication**, we mean a (new) operation \rightsquigarrow , with which \wedge forms a residuated pair. A question is when it is possible to introduce a Heyting implication \rightsquigarrow to a QSL \mathbf{L} in a conservative way.

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By combining our previous results, a sufficient condition for the conservativity of Heyting implication over a logic \mathbf{L} is given as follows. That is,

- 1 the schema (\wedge, \exists) holds in \mathbf{L} ,
- 2 all axioms of \mathbf{L} are preserved under complete ideal completions of every "algebra" for \mathbf{L} .

Relevant predicate logics **MQX** in Goldblatt (2009) are examples of these logics if we add moreover the axiom (\wedge, \exists) to them. They are shown by using preservation results.



R. Goldblatt, "Conservativity of Heyting implication over relevant quantification", Review of Symbolic Logic 2, 2009.

VI.1 Kripke-type semantics

Kripke-type semantics for substructural propositional logics are discussed in Ono-Komori (1985), K. Došen (1989,1990). A close connection between ideal completions and this semantics is stated already in HO 2003 for propositional logics.

For a given CRL \mathbf{A} , let \mathbf{S}_A be the reduct of the (lower) semilattice-ordered monoid. Then, for any propositional formula α , α is valid in the Kripke frame \mathbf{S}_A iff it is valid in the algebra $\mathbf{A}^J (= \wp(\mathbf{S}_A)_J)$. Hence

- a substructural propositional logic \mathbf{L} is Kripke-complete whenever the variety of \mathbf{FL}_e -algebras corresponding to \mathbf{L} is closed under ideal completions.

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- a substructural propositional logic \mathbf{L} is Kripke-complete whenever the variety of \mathbf{FL}_e -algebras corresponding to \mathbf{L} is closed under **ideal completions**.

This makes an interesting contrast with the following (also for modal logics).




- a superintuitionistic propositional logic \mathbf{L} is Kripke-complete whenever the variety of Heyting algebras corresponding to \mathbf{L} is closed under **canonical completions**.

VI.2 Kripke-Joyal semantics for QSLs

As a natural extension of this idea to semantics for QSLs, we can show a similar close connection between complete ideal completions and **Kripke-Joyal semantics**.

Let \mathbf{L} be a distributive substructural propositional logic. Then our preservation result tells us when the minimum predicate extension of \mathbf{L} with the axiom (\wedge, \exists) becomes Kripke-Joyal complete.

For further information, see:

-  J.L. Bell, "Cover schemes, frame-valued sets and their potential uses in spacetime physics", 2007.
-  R. Goldblatt, "A Kripke-Joyal semantics for noncommutative logic in quantales", Advances in Modal Logic 6, 2006.
-  HO, "Semantical analysis of predicate logics without the contraction rule", Studia Logica 44, 1985.