

Web-geometric approach to continuous triangular subnorms

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- 1 Definitions
- 2 Rectangles and relations
- 3 Associativity of local togmas
- 4 Corollaries
- 5 Triangular Subnorms

- *magma* $(M, *)$... a set M with an operation $*$: $M \times M \rightarrow M$
- *togma* $(M, *, \leq)$ (totally ordered magma)
 - $(M, *)$... magma
 - (M, \leq) ... chain
 - $\forall x, y, z \in M: \quad x \leq y \Rightarrow x * z \leq y * z$
- *monoid* $(M, *, 1)$
 - $(M, *)$... magma
 - $*$... associative
 - 1 ... neutral element
- *tomonoid* $(M, *, 1, \leq)$ (totally ordered monoid)
 - $(M, *, 1)$... monoid
 - (M, \leq) ... chain

Powerset togma:

- $(M, *, \leq) \dots$ togma
- for $X, Y \in \mathcal{P}(M), z \in M$:
 - $X * Y = \{x * y \mid x \in X, y \in Y\}$
 - $X/Y = \{z \mid \{z\} * Y \subseteq X\}$
 - $Y \setminus X = \{z \mid Y * \{z\} \subseteq X\}$

Local togma:

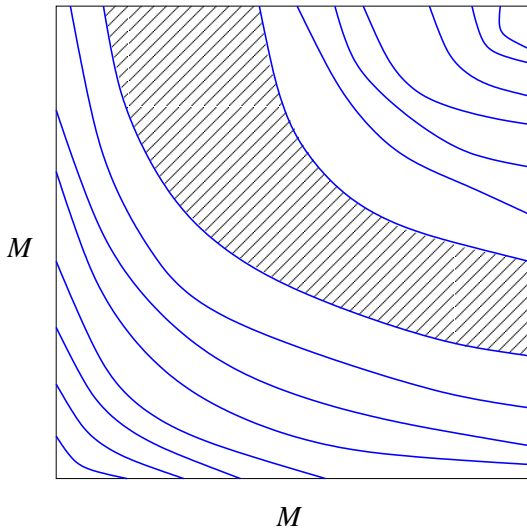
- $(L, \otimes, \leq) \dots$ *local togma* of M at $(u, v) \in M^2$ on $L \subseteq M$ if

$$\otimes: L \times L \rightarrow \mathcal{P}(L): (x, y) \mapsto (\{x\}/\{v\}) * (\{u\} \setminus \{y\})$$

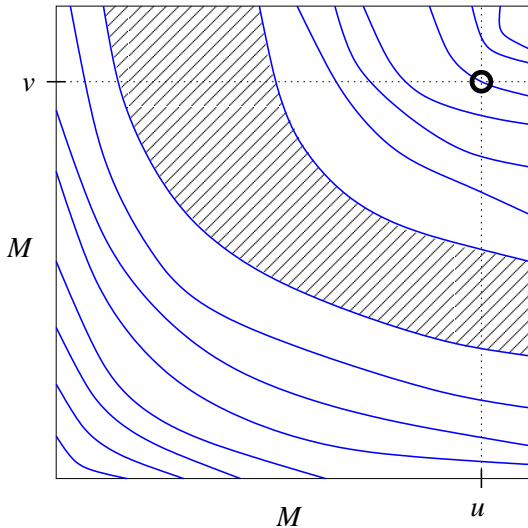
is a singleton for all $x, y \in L$.

- (in such a case, the singletons are identified with their elements
... we consider \otimes as a binary operation on L)

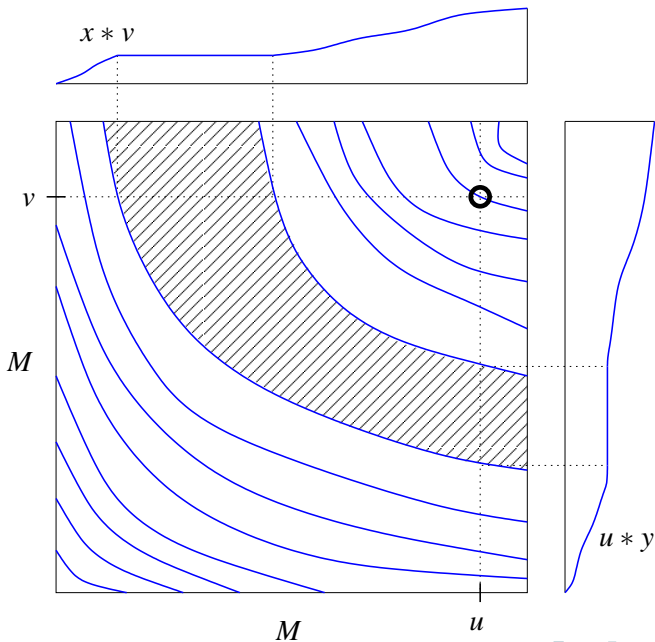
Computing $x \circledast y = (\{x\}/\{v\}) * (\{u\}\setminus\{y\}) \dots\dots\dots$ Example 1



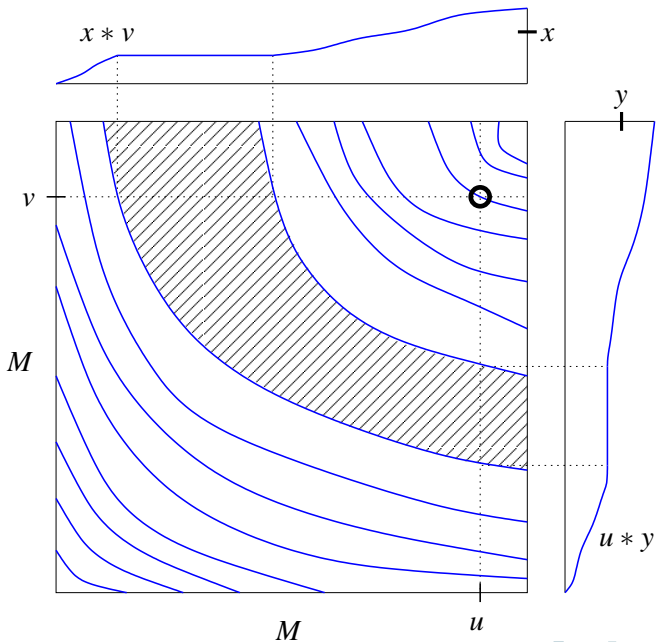
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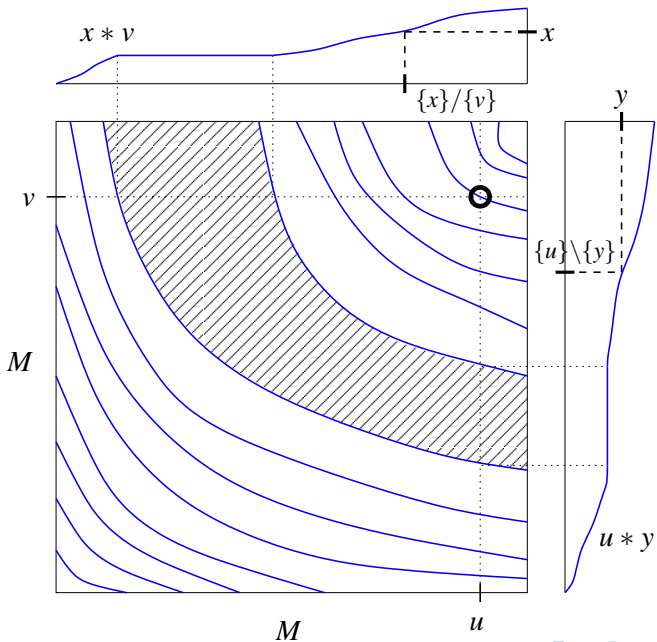
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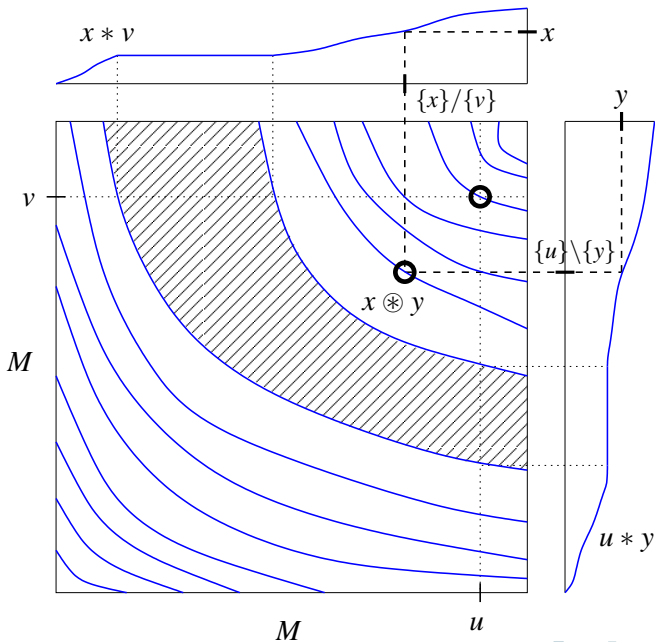
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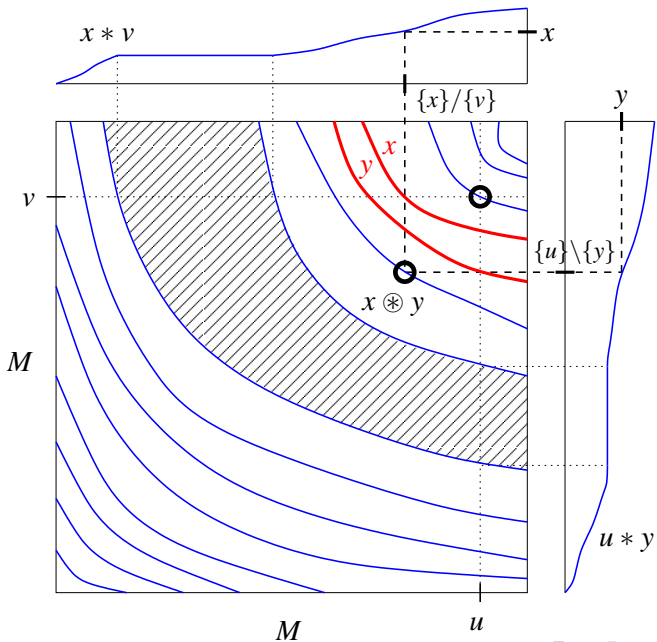
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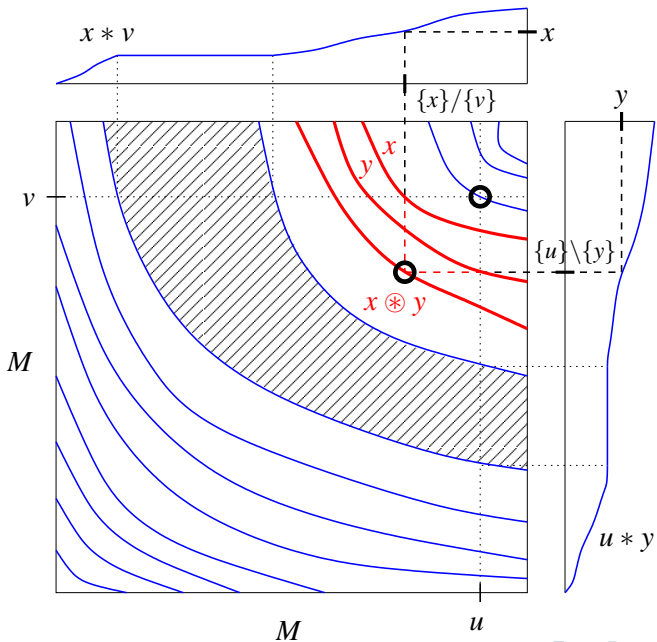
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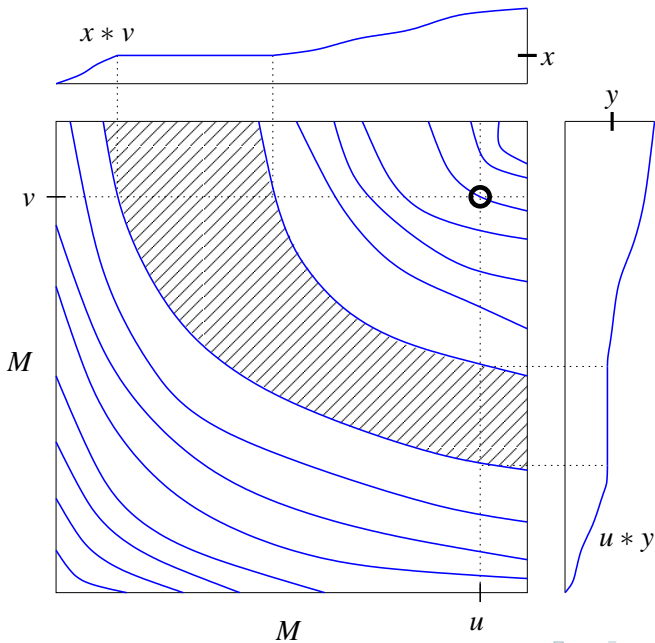
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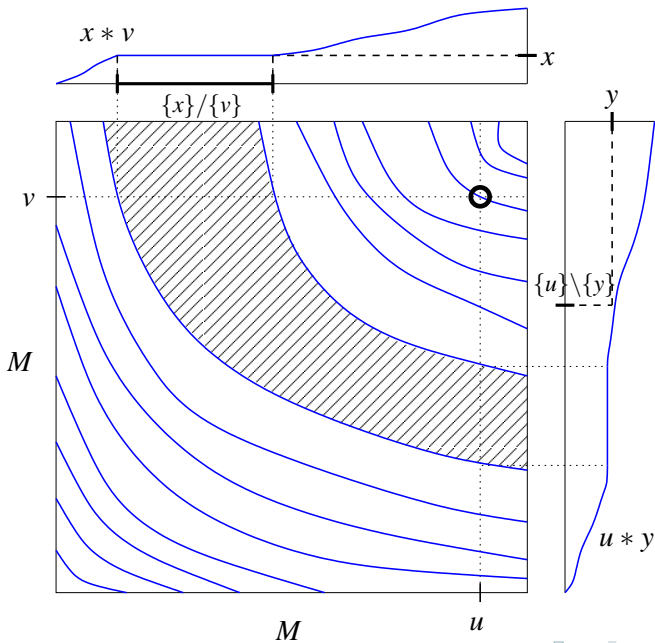
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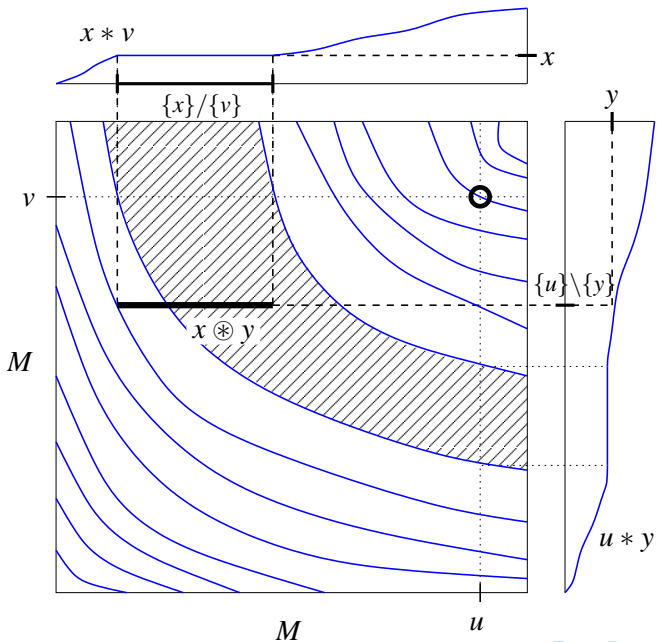
Computing $x \circledast y = (\{x\}/\{v\}) * (\{u\}\setminus\{y\}) \dots \dots \dots$ Example 2



Computing $x \circledast y = (\{x\}/\{v\}) * (\{u\} \setminus \{y\}) \dots \dots \dots$ Example 2



Computing $x \circledast y = (\{x\}/\{v\}) * (\{u\} \setminus \{y\})$ Example 2



Observations:

- $(M, *, \leq)$... togma
- (L, \otimes, \leq) ... *local togma* of M at $(u, v) \in M^2$ on $L \subseteq M$
- $e = u * v$ is the neutral element of \otimes
- $*$ is commutative $\Rightarrow \otimes$ is commutative
- $*$ is associative $\Rightarrow \otimes$ is associative

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Rectangles and relations

- $(M, *, \leq)$... toagma
- *rectangle* $P = \{a, b\} \times \{c, d\} \subset M^2$
- pair of rectangles P, R is (u, v) -local if:

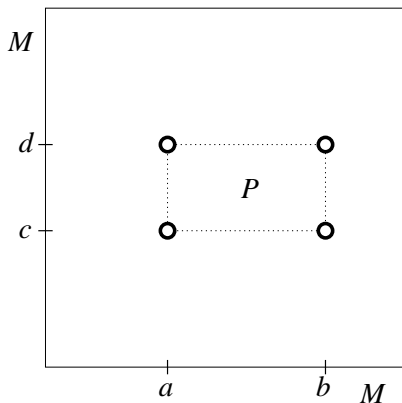
$$\begin{aligned}P &= \{a, b\} \times \{c, v\} \\R &= \{e, u\} \times \{g, h\}\end{aligned}$$

(where $a, b, c, e, g, h \in M$)

- P, R are *equivalent* ... $P \cong_* R$
 - functional values at the corresponding pairs of vertices are equal
- P, R are *strongly aligned* according to $(u, v) \in M^2$... $P \simeq_*^{uv} R$
 - functional values at the corresponding pairs of vertices are equal, except of the pair $(a, c), (e, g)$
- *intention*: to show that $P \simeq_*^{uv} R \Rightarrow P \cong_* R$ is related to the associativity of local tommas

Rectangles and relations

Rectangle $P = \{a, b\} \times \{c, d\} \subset M^2$



Rectangles and relations

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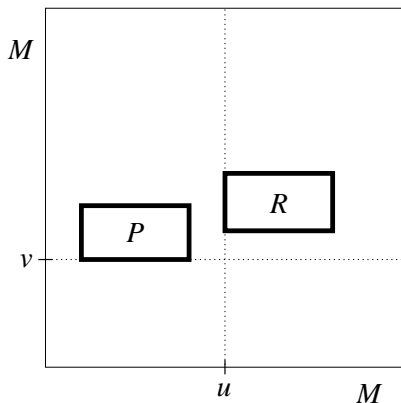
$$R = \{e, u\} \times \{g, h\}$$

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Rectangles and relations

Pair of (u, v) -local rectangles P, R



Rectangles and relations

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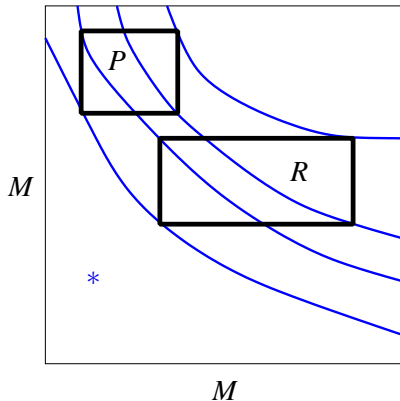
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Rectangles and relations

Equivalent rectangles $P, R \dots P \cong_* R$



Rectangles and relations

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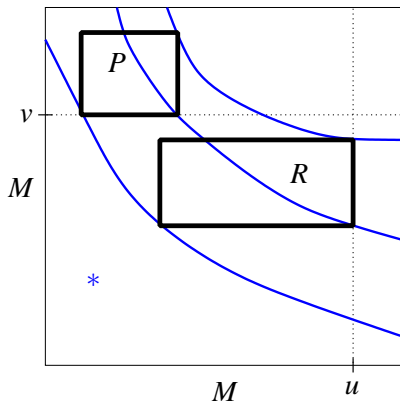
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Rectangles and relations

Rectangles P, R , strongly aligned according to $(u, v) \dots P \simeq_*^{uv} R$



Rectangles and relations

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- *rectangle* $P = \{a, b\} \times \{c, d\} \subset M^2$
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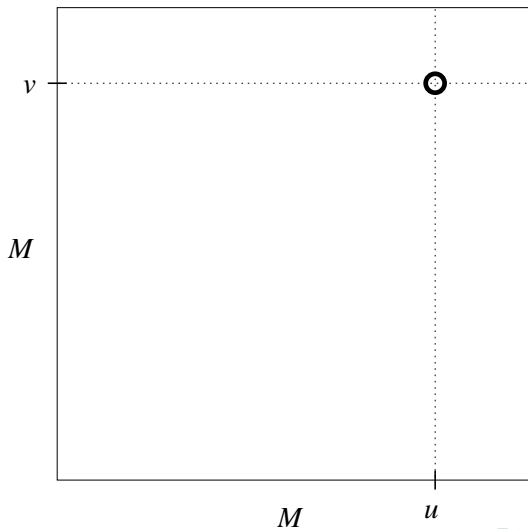
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Theorem:

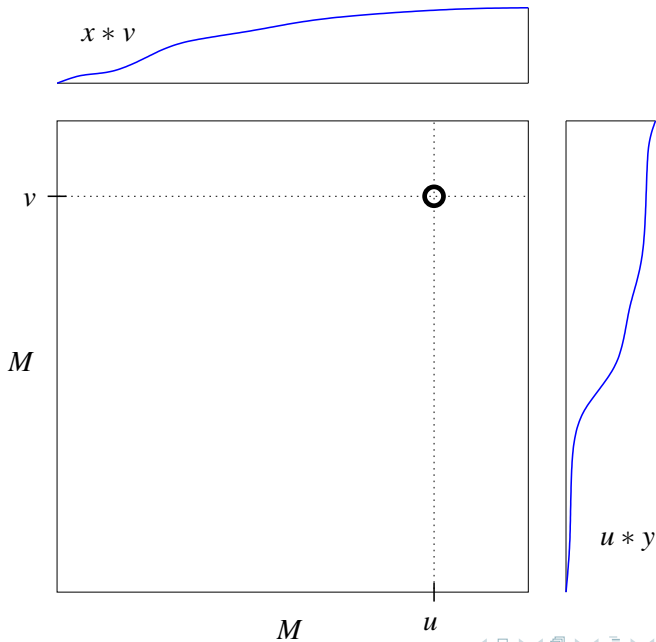
- $(M, *, \leq)$... toagma
- (L, \otimes, \leq) ... local toagma of $(M, *, \leq)$ at $(u, v) \in M^2$
- $\Omega = L/\{v\} \times \{u\} \setminus L$
- $e = u * v$
- Then the following statements are equivalent:
 - \otimes is associative ... and thus (L, \otimes, e, \leq) is a tomonoid
 - every pair of (u, v) -local rectangles $P, R \subset \Omega$ satisfies

$$P \simeq_*^{uv} R \Rightarrow P \cong_* R$$

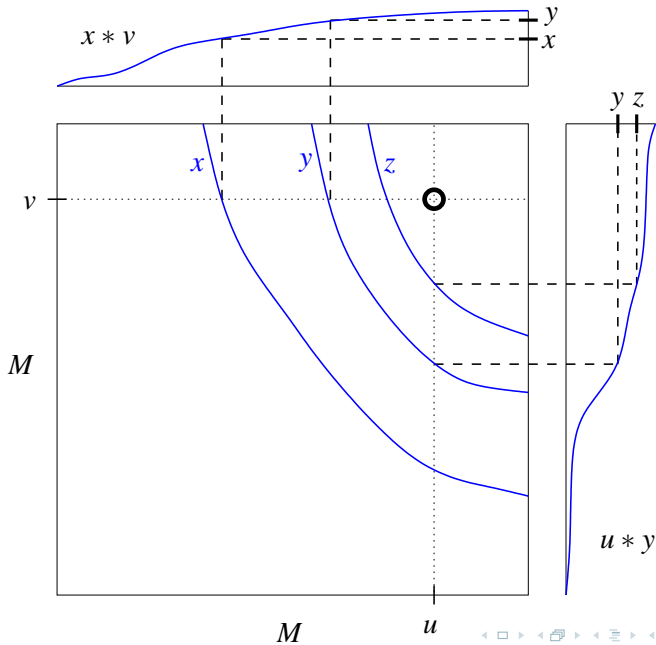
Proving: $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ $x \otimes y = (\{x\}/\{v\}) * (\{u\}\setminus\{y\})$



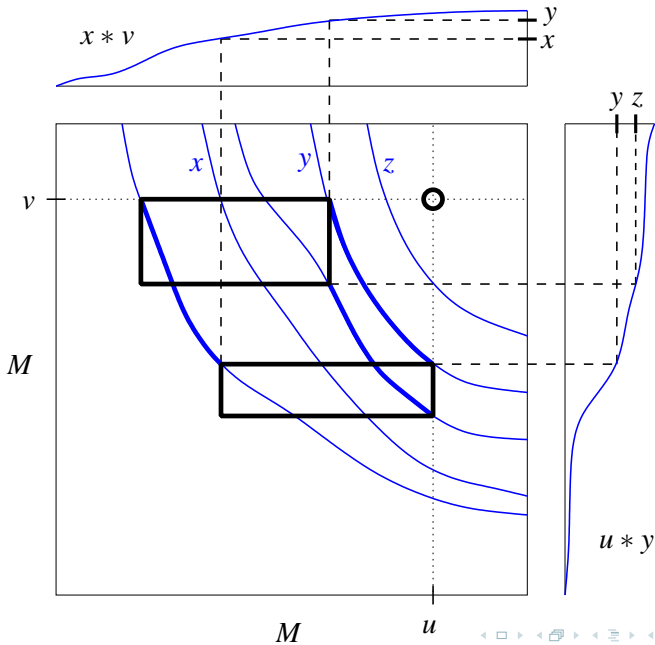
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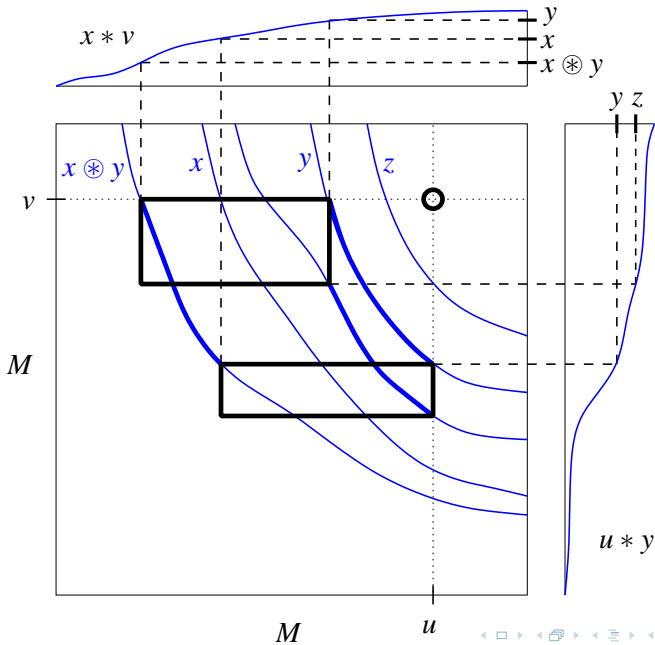
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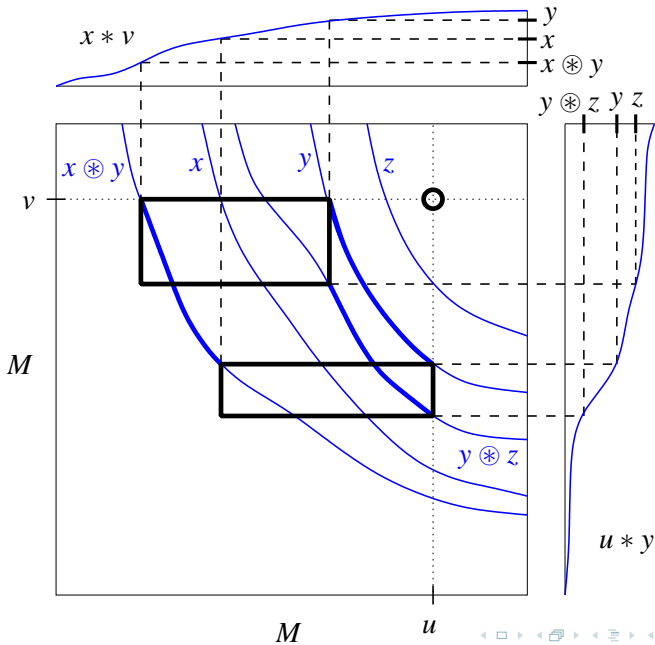
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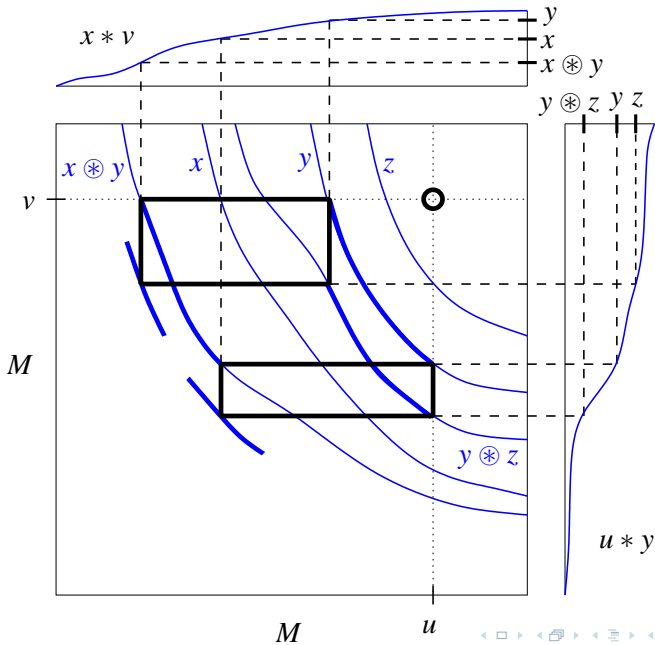
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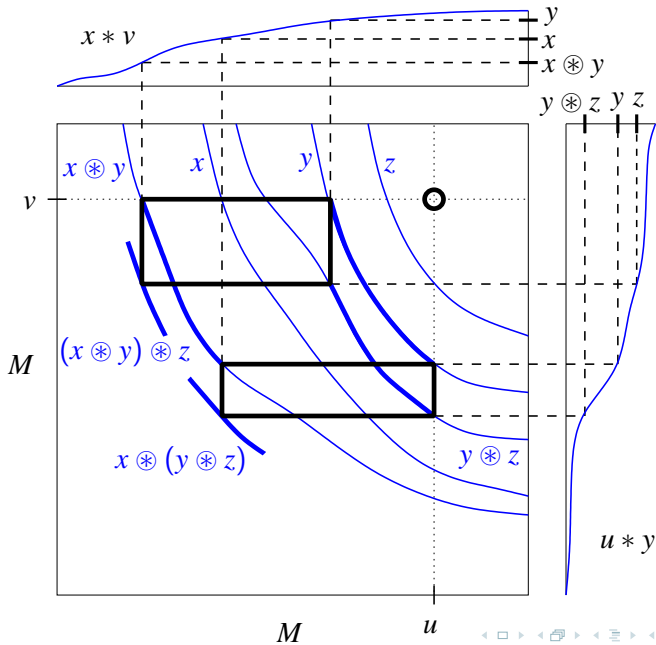
Proving: $(x \circledast y) \circledast z = x \circledast (y \circledast z)$ $x \circledast y = (\{x\} / \{v\}) * (\{u\} \setminus \{y\})$



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Proposition:

- $(M, *, 1, \leq)$... toagma with neutral element 1
- local toagma at $(1, 1)$ is actually the toagma $(M, *, 1, \leq)$ itself

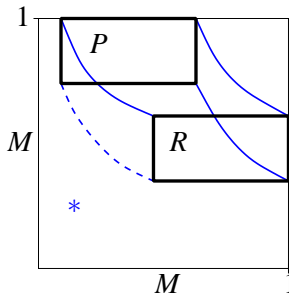
Corollary:

- $(M, *, 1, \leq)$ is tomonoid if and only if:

$$P \simeq_*^{11} R \Rightarrow P \cong_* R$$

for every pair of $(1, 1)$ -local rectangles P, R

Integral tomonoids:



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Triangular Subnorm

A mapping $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$

- commutative
 - associative
 - isotone with respect to \leq
 - $x * y \leq x \wedge y$ for all $x, y \in [0, 1]$
-
- left-continuous t-norms are ordinal sums of left-continuous t-subnorms
 - forcing 1 to be the unit element turns $*$ into a t-norm

Positive Cancellativity

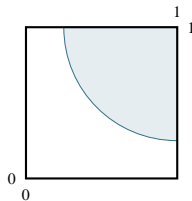
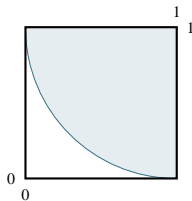
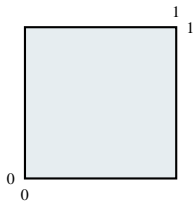
A binary operation $\bullet: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is **positively cancellative** (or **weakly cancellative**) if

$$a \bullet b = a \bullet c > 0 \quad \Rightarrow \quad b = c.$$

Support

For $\bullet: [0, 1] \times [0, 1] \rightarrow [0, 1]$ we define:

$$\text{supp } \bullet = \{(x, y) \in [0, 1] \times [0, 1] \mid x \bullet y > 0\}.$$

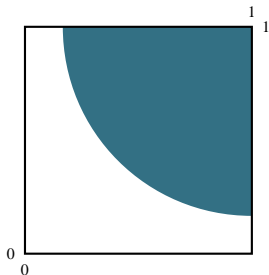


A. Mesiarová: *Special classes of t-norms*. Ph.D. Thesis.

Support Decomposition

Let $*$ be a t-subnorm. Denote:

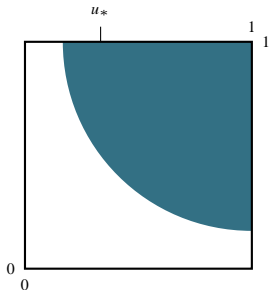
- $u_* = 1 * 1$
- $v_* = \sup\{z \in [0, 1] \mid 1 * z = 0\}$
- $s_* = \sup\{z \in [0, 1] \mid 1 * z = v_*\}$
- $U_* = \text{supp } * \setminus [u_*, 1[{}^2$
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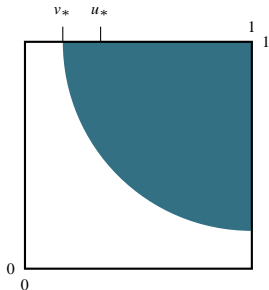
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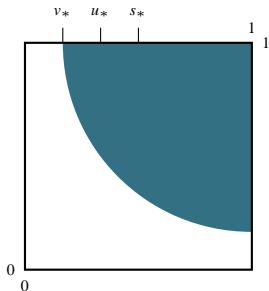
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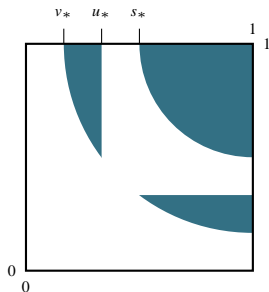
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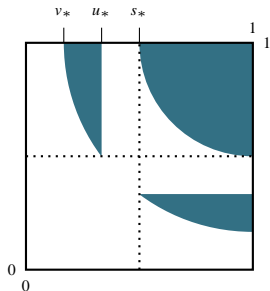
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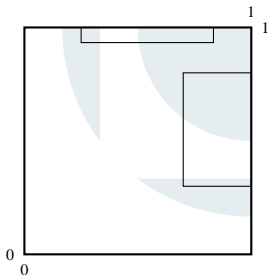
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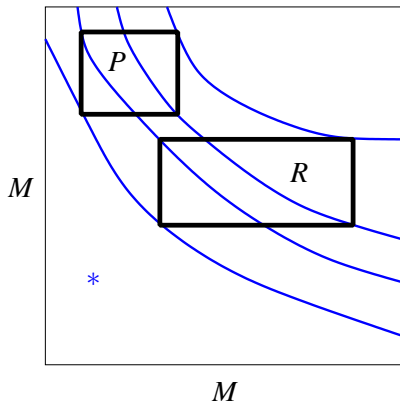
Web Geometry of Triangular Subnorms

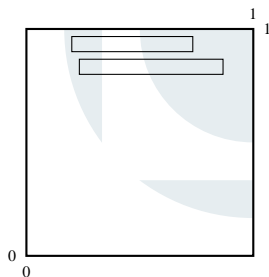


Proposition:

Given a t-subnorm $*$ the corresponding t-norm \bullet satisfies the Reidemeister condition for every pair of $(1, 1)$ -local rectangles.

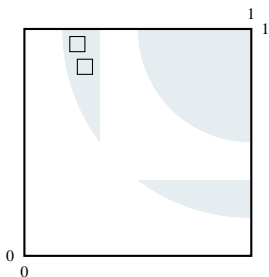
Weak Alignment





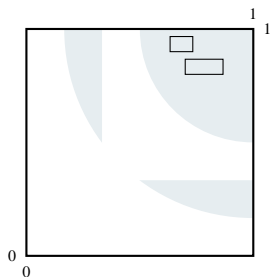
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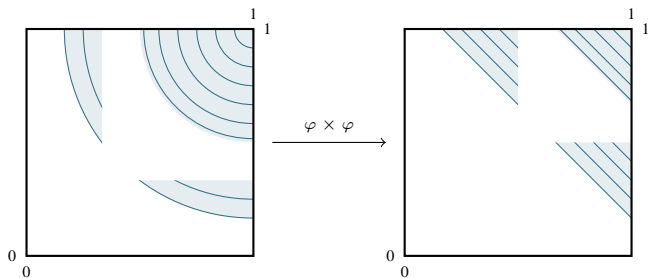


Theorem:

Given a t-subnorm $*$ the corresponding t-norm \bullet satisfies the Reidemeister condition for every pair of rectangles confined within the set $U_* \cup V_*$.

Corollary:

- there exists a transformation $\phi: [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1]$ such that $\phi = \varphi \times \varphi$ where $\varphi: [0, 1] \rightarrow [0, 1]$ is an increasing bijection
- moreover, the levelsets of $*$ push forward along ϕ to a system of parallel lines.



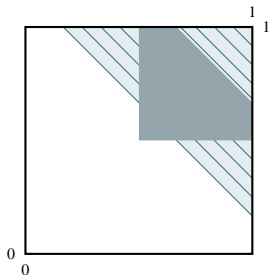
Definition:

A t-subnorm $*$ is said to be **Łukasiewicz-like** if there exists a nondecreasing mapping $\psi: [0, 1] \rightarrow [0, 1]$ with $\psi(0) = 0$ such that $* = \psi \circ \odot$ where \odot is the Łukasiewicz t-norm.

Definition:

- \mathcal{L} — the set of all Łukasiewicz-like t-subnorms
- \mathcal{L}^\pm — the set of all continuous, positively cancellative operations $\otimes: [0, 1] \times [0, 1] \rightarrow [0, 1]$ such that there exists a t-subnorm $* \in \mathcal{L}$ with

$$\otimes \upharpoonright_{U_\otimes \cup V_\otimes} = * \upharpoonright_{U_* \cup V_*} .$$



Theorem:

Continuous positively cancellative t-subnorms are either

- strict, or
- order-isomorphic images of t-subnorms from \mathcal{L}^{\pm} .