Compatible operations on residuated lattices

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Heyting frontal algebras

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A Heyting frontal algebra is an algebra $(\mathbf{H}, \tau)$, where $\mathbf{H} = (H, \wedge, \vee, \rightarrow, 0, 1)$ is a Heyting algebra and τ is an unary operator that satisfies the following equations:

(f1) $\tau(x \wedge y) = \tau(x) \wedge \tau(y)$,

(f2) $x \leq \tau(x)$,

(f3) $\tau(x) \leq y \vee (y \rightarrow x)$

Operation τ is called *frontal*.

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- By equations $(f1)$ and $(f2)$ we have that frontal operators are compatible functions.

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Heyting frontal algebras are the algebraic models of an augmentation of the Heyting propositional calculus by a particular modal operator, and they were studied mainly by Leo Esakia in the following article:

- The modalized Heyting calculus: a conservative modal extension of the Intuitionistic Logic. Journal of Applied Non-Classical Logics, vol 16-No.3-4, 349-366, 2006.

Heyting frontal algebras

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Other articles where frontal Heyting algebras were studied are the followings:

- Discrete Dualities for Heyting algebras with Operators. Orlawska, E. and Rewitzki I., Fundamenta Informaticae 81, 275-295, 2007.
- On frontal Heyting algebras. Sagastume M., Castiglioni J. L. and San Martín H.J., Reports on Mathematical Logic, vol 45, 201-224, 2010.

First example: the successor function

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Let **H** be a Heyting algebra and $S : H \to H$ the unary operation that satisfies the following equations:

(S1) $S(x) \to x \leq S(x)$,

(S2) $S(x) \leq y \vee (y \to x)$.

Function S is called *successor* and it is characterized by

$$S(x) = min\{y \in H : y \to x \leq y\}.$$

Second example: the gamma function

Second example: the gamma function

Let **H** be a Heyting algebra and $\gamma : H \to H$ the unary operation that satisfies the following equations:

(g1) $\neg\gamma(0) = 0$,

(g2) $\gamma(0) \leq (x \vee \neg x)$,

(g3) $\gamma(x) = x \vee \gamma(0)$.

This function is called gamma´s function and it is characterized by

$$\gamma(x) = min\ \{y \in H : \neg y \vee x \leq y\}.$$

Third example: function G

Third example: function G

Let **H** be a Heyting algebra and $G : H \to H$ the unary operation that satisfies the following equations:

(G1) $(G(x) \to x) \wedge \neg\neg x \leq x$,

(G2) $G(x) \leq y \vee ((y \to x) \wedge \neg\neg x)$.

This function is called Gabay´s function and it is characterized by

$$G(x) = min\ \{y \in H : (y \to x) \wedge \neg\neg x \leq y\}.$$

Relationship between the examples

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Caicedo and Cignoli prove in the article *An algebraic approach to intuitionistic connectives* (Journal of Symbolic Logic, 66, N^{o}4 ,1620-1636, 2001) the following facts:

- γ and G are definable in terms of S, since

$$\gamma(x) = x \vee S(0),\ G(x) = S(x) \wedge \neg\neg x.$$

  Function S is not definable from G or γ. Function G and γ are not mutually definable. However S is definable from G and γ as

$$S(x) = \gamma(x) \vee G(x).$$

  Besides these functions are not terms in the language of Heyting algebras.

Frontal residuated lattices

Frontal residuated lattices

Let $\mathbf{L} = \langle L, \wedge, \vee, ., e, \backslash, / \rangle$ be a residuated lattice (RL for short). We say that function $\tau : L \to L$ is a T*-left pre-frontal operator* ($T.l.p$ for short) if there is a binary term T in the language of residuated lattices such that for every Heyting algebra **H** we have that

$$T^{\mathbf{H}}(x, y) = y \to x$$

and for every $x, y \in L$ the following equations hold:

(l1) $\tau(x) \leq y \vee T(x, y)$,

(l2) $e \leq \tau(e)$,

(l3) $(x \backslash y) \wedge e \leq \tau(x) \backslash \tau(y)$.

Frontal residuated lattices

Frontal residuated lattices

If τ satisfies the additional equation

(I4) $\tau(x) \wedge \tau(y) \leq \tau(x \wedge y)$,

we say that τ is a *T-left frontal operator* ($T.l.f$ for short).

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Similarly we define T*-right pre-frontal operators* ($T.r.p$ for short) and T*-right frontal operators* ($T.r.f$ for short).

- If **H** is a Heyting algebra and $\tau : H \to H$ is a function then

  τ is a frontal operator iff τ is a $T.l.f$ iff τ is a $T.r.f$.

- $T.l.p$ and $T.r.p$ are compatible functions.

Generalized successor operators

Generalized successor operators

Let **L** be a RL. For every $y \in L$ we define $y^0 = e$ and for $k \geq 1$, $y^k = y^{k-1}.y$. Fix a natural number n. We define the unary functions $\overleftarrow{S}_n$ (n-left successor) and $\overrightarrow{S}_n$ (n-right successor) through the following equations:

$(LS1n)$ $\overleftarrow{S}_n(x)^n \backslash x \leq \overleftarrow{S}_n(x)$. $(LS2n)$ $\overleftarrow{S}_n(x) \leq y \vee (y^n \backslash x)$.

$(RS1n)$ $x / \overrightarrow{S}_n(x)^n \leq \overrightarrow{S}_n(x)$. $(RS2n)$ $\overrightarrow{S}_n(x) \leq y \vee (x / y^n)$.

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$(RS1n)$ $x / \overrightarrow{S}_n(x)^n \leq \overrightarrow{S}_n(x)$. $(RS2n)$ $\overrightarrow{S}_n(x) \leq y \vee (x / y^n)$.

- $\overleftarrow{S}_n$ is a $T.l.p$ taking $T(x,y) = y^n \backslash x$. Moreover, $\overleftarrow{S}_n(e) = e$.
- If the underlying lattice of **L** is distributive then $\overleftarrow{S}_n$ is a $T.l.f$.
- $\overleftarrow{S}_n$ is characterized by $\overleftarrow{S}_n(x) = min\{y \in L : y^n \backslash x \leq y\}$.

Generalized gamma operators

Generalized gamma operators

Let **L** be a RL with first element. We define the functions $\overleftarrow{\gamma_n} : L \to L$ (n-left gamma) and $\overrightarrow{\gamma_n} : L \to L$ (n-right gamma) through the following equations:

$(Lg1n)$ $\overleftarrow{\gamma_n}(0)^n \backslash 0 \leq \overleftarrow{\gamma_n}(0)$. $(Rg1n)$ $0/\overrightarrow{\gamma_n}(0)^n \leq \overrightarrow{\gamma_n}(0)$.

$(Lg2n)$ $\overleftarrow{\gamma_n}(0) \leq y \vee (y^n \backslash 0)$. $(Rg2n)$ $\overrightarrow{\gamma_n}(0) \leq y \vee (0/y^n)$.

$(Lg3n)$ $\overleftarrow{\gamma_n}(x) = x \vee \overleftarrow{\gamma_n}(0)$. $(Rg3n)$ $\overrightarrow{\gamma_n}(x) = x \vee \overrightarrow{\gamma_n}(0)$.

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- $\overleftarrow{\gamma_n}$ is a $T.l.p$ taking $T(x, y) = x \vee (y^n \backslash x)$.
- If the underlying lattice of **L** is distributive then $\overleftarrow{\gamma_n}$ is a $T.l.f$. Moreover, in this case $\overleftarrow{\gamma_n}$ preserves $\wedge$
- $\overleftarrow{\gamma_n}$ is characterized by $\overleftarrow{\gamma_n}(x) = min\{y \in L : (y^n \backslash 0) \vee x \leq y\}$.

Generalizated Gabbay’s operators

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Let **L** be a RL with first element. For every $x \in L$ we define $l(x) = x\backslash 0$, $r(x) = 0/x$ and the functions $\overleftarrow{G_n} : L \longrightarrow L$ (n-left G) and $\overrightarrow{G_n} : L \longrightarrow L$ (n-right G) through the following equations:

$(LG1n)$ $(\overleftarrow{G_n}(x)^n\backslash x) \wedge rl(x) \leq \overleftarrow{G_n}(x)$.
$(LG2n)$ $\overleftarrow{G_n}(x) \leq y \vee ((y^n\backslash x) \wedge rl(x))$.

$(RG1n)$ $(x\backslash\overrightarrow{G_n}(x)^n) \wedge lr(x) \leq \overrightarrow{G_n}(x)$.
$(RG2n)$ $\overrightarrow{G_n}(x) \leq y \vee ((x/y^n) \wedge lr(x))$.

Generalizated Gabbay's operators

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- $\overleftarrow{G_n}$ is a $T.l.p$ taking $T(x,y) = y^n\backslash x$. Moreover, $\overleftarrow{G_n}(e) = e$.
- $\overleftarrow{G_n}$ is characterized by
  $\overleftarrow{G_n}(x) = min\{y \in L : (y^n\backslash x) \wedge rl(x) \leq y\}$.

An example

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Let L be the chain $\{0, a, b, 1\}$ with the following operations:

$.$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	a	b	b
1	0	a	1	1

$/$	0	a	b	1
0	1	a	a	0
a	1	1	a	a
b	1	1	b	b
1	1	1	1	1

$\backslash$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $\mathbf{L} = \langle L, \wedge, \vee, ., 1, \backslash, / \rangle$ is a RL and we have the following tables:

Left side: $n \geq 1$ - Right side $n \geq 2$

x	$\overleftarrow{S}_n(x)$	$\overrightarrow{S}_1(x)$	$\overleftarrow{\gamma}_n(x)$	$\overrightarrow{\gamma}_1(x)$
0	b	a	b	a
a	b	b	b	a
b	1	b	b	b
1	1	1	1	1

x	$\overrightarrow{S}_n(x)$	$\overrightarrow{\gamma}_n(x)$
0	b	b
a	b	b
b	b	b
1	1	1

x	$rl(x)$	$lr(x)$	$\overleftarrow{G}_n(x)$	$\overrightarrow{G}_n(x)$
0	0	0	0	0
a	a	b	a	b
b	1	0	1	0
1	1	0	1	0

Operations given by equations

Let V be a variety of algebras of type F and let $\epsilon(C)$ be a set of identities of type $F \cup C$ where C is a family of new function symbols. We say that $\epsilon(C)$ defines *implicitly* C, if in each algebra $A \in V$ there is at most one family $\{f_H : H^n \to H\}_{f \in C}$ such that $(A, f_A)_{f \in C}$ satisfies the universal clousure of the equations in $\epsilon(C)$ (in this case we say that each f is given by equations).

For example, n-left successor, n-right successor, n-left gamma, n-right gamma, n-left G and n-right G are given by equations.

Operations given by the min Operator

Operations given by the min Operator

In the article *Compatible operations on commutative residuated lattices* (JANCL, vol 18, 413-425, 2008) of Castiglioni, Menni and Sagastume, a condition is given on a function $P(x, y)$ in a commutative residuated lattice L that implies that the function $x \mapsto min\{y \in L : P(x, y) \leq y\}$ is equational and compatible when defined.
Inspireted by this article one can ask whether conditions on functions $P(x, y)$ and $Q(x, y)$ in a residuated lattice L imply that the function

$$x \mapsto min\{y \in L : P(x, y) \leq Q(x, y)\}$$

is equational and compatible when defined.

Operations given by the min Operator

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Let $P : L \times L \to L$ and $Q : L \times L \to L$ be two functions on a partial order with binary suprema $(L, \vee)$ such that

1. $y \geq z$ implies $P(x, y) \leq P(x, z)$,
2. $P(x, P(x, y) \vee Q(x, y)) \leq Q(x, P(x, y) \vee Q(x, y))$,
3. $Q(x, y) \leq y$,

In this case we will say that P is Q-pre-compatible.

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2. $P(x, P(x, y) \vee Q(x, y)) \leq Q(x, P(x, y) \vee Q(x, y))$,
3. $Q(x, y) \leq y$,

In this case we will say that P is Q-pre-compatible.
If P is Q-pre-compatible we denote by $f_{(P,Q)} : L \to L$ the function that to each x in L assigns the least element of the set $E_{(P,Q)}(x)$, where

$$E_{(P,Q)}(x) := \{y \in L : P(x, y) \leq Q(x, y)\}.$$

Operations given by the min Operator

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- Let $P : L \times L \to L$ and $Q : L \times L \to L$ be binary functions on a partial order with binary suprema $(L, \vee)$ such that P is Q-pre-compatible. Then the following are equivalent:

  1. There exists the function $f_{(P,Q)}$.

  2. There exists a function $g : L \to L$ such that

  (PQ1) $P(x, g(x)) \leq Q(x, g(x))$

  (PQ2) $g(x) \leq P(x, y) \vee Q(x, y)$.

  Moreover, in this case $g = f_{(P,Q)}$.

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     (PQ2) $g(x) \leq P(x, y) \vee Q(x, y)$.

     Moreover, in this case $g = f_{(P,Q)}$.

Let L be a RL.

- If P is compatible in the first variable then g is compatible.

Operations given by the min Operator

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The examples of $T.l.p$ and $T.r.p$ given before fit in this context too. Take $Q(x, y) = y$ and n be a natural number.

$P(x,y)$	$g(x)$
x/y^n	$\overrightarrow{S}_n$
$y^n\backslash x$	$\overleftarrow{S}_n$
$(0/y^n) \vee x$	$\overrightarrow{\gamma}_n$
$x \vee (y^n\backslash 0)$	$\overleftarrow{\gamma}_n$
$(x/y^n) \wedge lr(x)$	$\overrightarrow{G}_n$
$(y^n\backslash x) \wedge rl(x)$	$\overleftarrow{G}_n$

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$(y^n \backslash x) \wedge rl(x)$	$\overleftarrow{G}_n$

Operations given by the min Operator

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Let **L** be a residuated lattice such that the underlying lattice is distributive, and let n be a natural number. We consider

$$Q(x, y) = y \wedge e.$$

If we consider

$$P(x, y) = e \wedge (x/(y \wedge e)^n)$$

or

$$P(x, y) = e \wedge ((y \wedge e)^n \backslash x)$$

then P is Q-pre-compatible.

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Final Remark

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- Let L be a finite residuated lattice and $\{f_i\}_i$ a family of compatible functions given by equations such that $(L, \{f_i\}_i)$ do not have proper subalgebras. Then the variety $V(L, \{f_i\}_i)$ is affine complete.

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It follows from the facts that RL is locally affine complete and congruence distributive.

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It follows from the facts that RL is locally affine complete and congruence distributive.

For instance, if we consider the example seen of the chain of four elements then the following varieties are under the hypothesis of this proposition: $V(L, 0, \overrightarrow{S_1})$, $V(L, 0, \overrightarrow{\gamma_1})$, $V(L, 0, \overrightarrow{G_n}, \overleftarrow{G_n})$, $V(L, 0, \overleftarrow{G_n}, \overleftarrow{\gamma_n})$.