Paraconsistent Fuzzy Logic - A Review

Esko Turunen
These results are published in FSS and LNCS (2009)
Co–authors M. Öztürk and A. Tsoukiás

September 8, 2010

Stanford Encyclopedia of Philosophy: The contemporary logical orthodoxy has it that, from contradictory premises, anything can be inferred. To be more precise, let $\models$ be a relation of logical consequence, defined either semantically or proof–theoretically. Call $\models$ explosive if it validates $\{A, \neg A\} \models B$ for every A and B (ex contradictione quodlibet).

The major motivation behind paraconsistent logic is to challenge this orthodoxy. A logical consequence relation is said to be paraconsistent if it is not explosive. Thus, if $\models$ is paraconsistent, then even if we are in certain circumstances where the available information is inconsistent, the inference relation does not explode into triviality. Therefore, paraconsistent logic accommodates inconsistency in a sensible manner that treats inconsistent information as informative.

We meet paraconsistent reasoning e.g. in

Legal proceedings Assume there is an accused person who does not confess. Then the verdict, guilty or not guilty, is to be done on the basis of circumstantial evidence. The defence counsel, of course, presents evidence for the innocence of the accused person, while the prosecutor present edivence agaist the accused. Even if such information is contradictory a verdict can be given.

Decision–making in the European Union A general principle for a new directive to become effective is that there is big enough majority supporting the directive and small enough minority agaist the directive. This, of course, couses political volte–faces.

Our motivation in developing paraconsistent Pavelka style is to offer formal tools to handle such situations.

In Łukasiewicz infinite valued propositional logic there are four axioms and Modus Ponens as a rule of inference. Formulae are valuated on the real unit interval $[0,1]$. Unlike in classical logic,

$$\not\models \alpha \Rightarrow \alpha \& \alpha.$$

In 1979 Pavelka extended Łukasiewicz logic by adding truth constants: they generalize the symbols $\bot$ and $\top$ of classical logic. For each real in $[0,1]$ there is a truth constant in the formal language $\mathcal{F}$. Unfortunately the language is no more countable (this problem was solved by Hájek who showed that it is enough to have a truth constant for each rational in $[0,1]$). Pavelka introduced a formal fuzzy theory and the concepts partial tautology and partial proof, he also proved that they coincide.

Most remarkable is that everything that can be done in Boolean logic can be done in Łukasiewicz-Pavelka graded logic, too.

An MV–algebra is called complete if the underlying lattice is a complete lattice.

Definition

A complete MV-algebra **L** is injective if, for any $a \in L$ and any natural number n, there is an element $b \in L$, called the n–divisor of a, such that $nb = \underbrace{b \oplus \cdots \oplus b}_{n \text{ times}} = a$ and $(a^* \oplus (n-1)b)^* = b$.

All n–divisors are unique (Kukkurainen & Turunen 2002). The Łukasiewicz structure $\mathcal{L}$ is an injective MV–algebra, moreover, a finite product of injective MV–algebras is an injective MV–algebra. Gluschankof (1992) and Di Nola & Sessa (1995) have characterized injective MV–algebras.

Proved by Turunen in 1995, Pavelka's program is realizable in any injective MV–algebra L. Thus, assume a language $\mathcal{F}$ of sentential logic with truth constants is given.
Any mapping $v : \mathcal{F}_a \mapsto L$ such that $v(\mathbf{a}) = a$ for all truth constants **a** extends into $\mathcal{F}$ by $v(\alpha \texttt{ imp } \beta) = v(\alpha) \rightarrow v(\beta)$ and $v(\alpha \texttt{ and } \beta) = v(\alpha) \odot v(\beta)$. Such mappings v are called valuations. The degree of tautology is

$$\mathcal{C}^{sem}(\alpha) = \bigwedge\{v(\alpha) \mid v \text{ is a valuation }\}.$$

Fix a fuzzy set $T \subseteq \mathcal{F}$ of wffs and consider valuations v such that $T(\alpha) \leq v(\alpha)$ for all wffs α. If such a valuation v exists, the T is called satisfiable. We say that T is a fuzzy theory and formulae α such that $T(\alpha) \neq 0$ are the *non–logical axioms* of the fuzzy theory T. Then we consider values

$$\mathcal{C}^{sem}(T)(\alpha) = \bigwedge\{v(\alpha) \mid v \text{ is a valuation, } v \text{ satisfies } T\}.$$

There are eleven logical axioms denoted by a set $\mathtt{A}$. A fuzzy rule of inference is a scheme

$$\frac{\alpha_1, \cdots, \alpha_n}{r^{\mathtt{syn}}(\alpha_1, \cdots, \alpha_n)} \quad , \quad \frac{a_1, \cdots, a_n}{r^{\mathtt{sem}}(a_1, \cdots, a_n)},$$

where the wffs $\alpha_1, \cdots, \alpha_n$ are premises and the wff $r^{\mathtt{syn}}(\alpha_1, \cdots, \alpha_n)$ is the conclusion. The values $a_1, \cdots, a_n$ and $r^{\mathtt{sem}}(a_1, \cdots, a_n) \in L$ are the corresponding truth values. The mappings $L^n \mapsto L$ are semi–continuous, i.e.

$$r^{\mathtt{sem}}(a_1, \cdots, \bigvee_{j\in\Gamma} a_{k_j}, \cdots, a_n) = \bigvee_{j\in\Gamma} r^{\mathtt{sem}}(a_1, \cdots, a_{k_j}, \cdots, a_n)$$

holds for all $1 \leq k \leq n$. Moreover, fuzzy rules are required to be sound in a sense that

$$r^{\mathtt{sem}}(v(\alpha_1), \cdots, v(\alpha_n)) \leq v(r^{\mathtt{syn}}(\alpha_1, \cdots, \alpha_n))$$

holds for all valuations v.

The following are examples of fuzzy rules of inference, denoted by a set R:

Generalized Modus Ponens :

$$\frac{\alpha, \alpha \ \mathtt{imp}\ \beta}{\beta} \quad , \quad \frac{a, b}{a \odot b}$$

a–Lifting rules :

$$\frac{\alpha}{\mathbf{a}\ \mathtt{imp}\ \alpha} \quad , \quad \frac{b}{a \to b}$$

where **a** is an inner truth value.

Rule of Bold Conjunction:

$$\frac{\alpha, \beta}{\alpha\ \mathtt{and}\ \beta} \quad , \quad \frac{A, B}{A \odot B}$$

Proved by Turunen in 1997, any classical rule of inference has a sound counterpart in Pavelka logic!

A meta proof w of a wff α in a fuzzy theory T is a finite sequence

$$\begin{array}{ccc} \alpha_1 & , & a_1 \\ \vdots & & \vdots \\ \alpha_m & , & a_m \end{array}$$

where
(i) $\alpha_m = \alpha$,
(ii) for each i, $1 \leq i \leq m$, α_i is a logical axiom, or is a non–logical axiom, or there is a fuzzy rule of inference in $\mathtt{R}$ and wff formulae $\alpha_{i_1}, \cdots, \alpha_{i_n}$ with $i_1, \cdots, i_n < i$ such that $\alpha_i = r^{\mathtt{syn}}(\alpha_{i_1}, \cdots, \alpha_{i_n})$,
(iii) for each i, $1 \leq i \leq m$, the value $a_i \in L$ is given by

$$a_i = \begin{cases} a & \text{if } \alpha_i \text{ is the axiom } \mathbf{a} \\ 1 & \text{if } \alpha_i \text{ is in } \mathtt{A} \\ T(\alpha_i) & \text{if } \alpha_i \text{ is a non–logical axiom} \\ r^{\mathtt{sem}}(a_{i_1}, \cdots, a_{i_n}) & \text{if } \alpha_i = r^{\mathtt{syn}}(\alpha_{i_1}, \cdots, \alpha_{i_n}) \end{cases}$$

The value a_m is called the degree of the meta proof w.

Since a wff α may have various meta proofs with different degrees, we define the degree of deduction of a formula α to be the supremum of all such values, i.e.,

$$\mathcal{C}^{syn}(T)(\alpha) = \bigvee\{a_m |\ w \text{ is a meta proof for } \alpha \text{ in } T\}.$$

A fuzzy theory T is consistent if $\mathcal{C}^{sem}(T)(\mathbf{a}) = a$ for all inner truth values $\mathbf{a}$. Any satisfiable fuzzy theory is consistent.

Theorem (Completeness of Pavelka style sentential logic)

In consistent fuzzy theories T, $\mathcal{C}^{sem}(T)(\alpha) = \mathcal{C}^{syn}(T)(\alpha)$, $\alpha \in \mathcal{F}$.

Thus, in Pavelka style fuzzy sentential logic we may talk about tautologies of a degree a and theorems of a degree a for all truth values $a \in L$, and these concepts coincide.

In 1977 Belnap introduced four possible values associated with a formula α in first order logic. They are `(what is told to be)` `true`, `false`, `contradictory` and `unknown`:

1. if there is evidence for α and no evidence against α, then α obtains the value `true`
2. if there is no evidence for α and evidence against α, then α obtains the value `false`
3. a value `contradictory` corresponds to a situation where there is simultaneously evidence for α and against α and, finally,
4. α is labeled by value `unknown` if there is no evidence for α nor evidence against α.

More formally, the values are associated with ordered couples $\langle 1, 0 \rangle$, $\langle 0, 1 \rangle$, $\langle 1, 1 \rangle$ and $\langle 0, 0 \rangle$, respectively.

In 1998, 2007, Perny, Tsoukias and Özturk imposed - being unaware of MV–algebras – a continuous valued extension of Belnap's logic. Given an ordered couple $\langle B(\alpha), B(\neg\alpha)\rangle$, graded values are to be computed via

$$\begin{aligned} t(\alpha) &= \min\{B(\alpha), 1 - B(\neg\alpha)\}, && (1)\\ k(\alpha) &= \max\{B(\alpha) + B(\neg\alpha) - 1, 0\}, && (2)\\ u(\alpha) &= \max\{1 - B(\alpha) - B(\neg\alpha), 0\}, && (3)\\ f(\alpha) &= \min\{1 - B(\alpha), B(\neg\alpha)\}. && (4)\end{aligned}$$

The intuitive meaning of $B(\alpha)$ and $B(\neg\alpha)$ is the degree of evidence for α and against α, respectively. Moreover, the set of 2×2 matrices of a form

$$\begin{bmatrix} f(\alpha) & k(\alpha) \\ u(\alpha) & t(\alpha) \end{bmatrix}$$

is denoted by $\mathcal{M}$. However, assuming a Boolean structure in $\mathcal{M}$ leads to anomalies.

Belnap's ideas can be extended to a Pavelka style fuzzy sentential logic.

Let $\mathbf{L} = \langle L, \oplus, ^*, \mathbf{0} \rangle$ be an MV–algebra. The product set $L \times L$ can be equipped with an MV–structure by setting

$$\langle a, b \rangle \otimes \langle c, d \rangle = \langle a \oplus c, b \odot d \rangle, \tag{5}$$

$$\langle a, b \rangle^{\perp} = \langle a^*, b^* \rangle, \tag{6}$$

$$\overline{\mathbf{0}} = \langle \mathbf{0}, \mathbf{1} \rangle \tag{7}$$

for each ordered couple $\langle a, b \rangle, \langle c, d \rangle \in L \times L$. The order on $L \times L$ is defined via

$$\langle a, b \rangle \leq \langle c, d \rangle \text{ if and only if } a \leq c, d \leq b, \tag{8}$$

The lattice operations are defined by

$$\langle a, b\rangle \vee \langle c, d\rangle = \langle a \vee c, b \wedge d\rangle, \tag{9}$$
$$\langle a, b\rangle \wedge \langle c, d\rangle = \langle a \wedge c, b \vee d\rangle, \tag{10}$$

and an adjoin couple $\langle \star, \mapsto\rangle$ by

$$\langle a, b\rangle \star \langle c, d\rangle = \langle a \odot c, b \oplus d\rangle, \tag{11}$$
$$\langle a, b\rangle \mapsto \langle c, d\rangle = \langle a \to c, (d \to b)^*\rangle. \tag{12}$$

Definition

Given an MV-algebra **L**, denote the structure described via (5) - (12) by $\mathbf{L}_{EC}$ and call it the MV–algebra of evidence couples induced by **L**. Moreover, denote

$$\mathcal{M} = \left\{ \begin{bmatrix} a^* \wedge b & a \odot b \\ a^* \odot b^* & a \wedge b^* \end{bmatrix} | \langle a, b \rangle \in L \times L \right\}$$

and call it the set of evidence matrices induced by evidence couples.

Then we have

Theorem

There is a one–to–one correspondence between $L \times L$ *and* $\mathcal{M}$*: if* $A, B \in \mathcal{M}$ *are two evidence matrices induced by evidence couples* $\langle a, b \rangle$ *and* $\langle x, y \rangle$*, respectively, then* $A = B$ *if and only if* $a = x$ *and* $b = y$*.*

Next we observe that the MV–structure descends from $\mathbf{L}_{EC}$ to $\mathcal{M}$ in a natural way: if $A, B \in \mathcal{M}$ are two evidence matrices induced by evidence couples $\langle a, b\rangle$ and $\langle x, y\rangle$, respectively, then the evidence couple $\langle a \oplus x, b \odot y\rangle$ induces an evidence matrix

$$C = \left[\begin{array}{cc} (a\oplus x)^* \wedge (b \odot y) & (a \oplus x) \odot (b \odot y) \\ (a\oplus x)^* \odot (b\odot y)^* & (a\oplus x) \wedge (b \odot y)^* \end{array}\right].$$

Thus, we may define a binary operation $\bigoplus$ on $\mathcal{M}$ by

$$\left[\begin{array}{cc} a^* \wedge b & a \odot b \\ a^* \odot b^* & a \wedge b^* \end{array}\right] \bigoplus \left[\begin{array}{cc} x^* \wedge y & x \odot y \\ x^* \odot y^* & x \wedge y^* \end{array}\right] = C.$$

Similarly, if $A \in \mathcal{M}$ is an evidence matrix induced by an evidence couple $\langle a, b\rangle$, then the evidence couple $\langle a^*, b^*\rangle$ induces an evidence matrix

$$A^{\perp} = \begin{bmatrix} a \wedge b^* & a^* \odot b^* \\ a \odot b & a^* \wedge b \end{bmatrix}.$$

In particular, the evidence couple $\langle \mathbf{0}, \mathbf{1}\rangle$ induces the following evidence matrix

$$F = \begin{bmatrix} \mathbf{0}^* \wedge \mathbf{1} & \mathbf{0} \odot \mathbf{1} \\ \mathbf{0}^* \odot \mathbf{1}^* & \mathbf{0} \wedge \mathbf{1}^* \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Theorem

Let **L** *be an MV–algebra. The structure* $\mathcal{M} = \langle \mathcal{M}, \bigoplus, ^{\perp}, F\rangle$ *as defined above is an MV-algebra (called the MV–algebra of evidence matrices).*

Our main algebraic result is the following

Theorem

L *is an injective MV–algebra if, and only if the corresponding MV–algebra of evidence matrices* $\mathcal{M}$ *is an injective MV–algebra.*

A immediate consequence is that, starting from an injective MV–algebra **L**, the corresponding $\mathcal{M}$–valued sentential logic is a sound and complete logic in Pavelka sense.

Recall a four–fold table from the GUHA theory

	ψ	$\neg\psi$
ϕ	a	b
$\neg\phi$	c	d

A statement connecting two attributes ϕ and ψ by basic double implicational quantifier is supported by the data or is TRUE if

$$a \geq n \text{ and } \frac{a}{a+b+c} \geq p,$$

where $n \in \mathcal{N}$ and $p \in (0,1]$ are parameters given by user.

In practical data mining it happens that indifferent cases rule over interesting cases, i.e. value d in a four–fold contingency table is much bigger that values a, b, c. However, even in such cases it is useful to look for statements Φ such that the truth value of Φ is, say at least $k(>1)$ times bigger than the falsehood of Φ, i.e. $\alpha \geq k\beta$, which is equivalent to $a \geq k(b+c)$. On the other hand such a statement Φ is stamped by label supported by the data if

$$\tfrac{a}{a+b+c} \geq p \text{ iff } a \geq \tfrac{p}{1-p}(b+c).$$

This means $k = \frac{p}{1-p}$, $p \neq 1$, or equivalently $p = \frac{k}{k+1}$. We have

Theorem

Given a data, all statements Φ such that the truth value of Φ is at least $k(>1)$ times bigger than the falsehood of Φ in the sense of paraconsistent logic, can be found by using basic double implicational quantifier and setting $p = \frac{k}{k+1}$.

OBUDA UNIVERSITY
TAMPEREEN TEKNILLINEN YLIOPISTO
UNIVERSIDADE DE COIMBRA
UNIVERSITE PARIS DAUPHINE
UNIVERSIDAD REY JUAN CARLOS
HABBERTEC

VRTUOSI
VIRTUAL MOBILITY IN DECISION SCIENCES
http://www.vrtuosi.com

VRTUOSI – open access course in decision theory starting autumn 2010, contact:
http://www.vrtuosi.com/
contains a detailed introduction to GUHA