Toward a model construction of H

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Outline

- **Title:** Toward a ~~forcing~~ model construction of $\mathbf{H}$
- **Future goal:** to construct a natural Tarskian model of $\mathbf{H}$.
  - Only known model of $\mathbf{H}$ is by a proof theoretic way [Whi79] (infinite set of sentences whose truth value are $\mathbf{1}$),
  - Since $\forall Ł$ is defined by its models (or by its truth functions) by one's nature, treating $\mathbf{H}$ via models should be a more natural way.
- **In this talk**, We would like to give a basic idea of forcing construction, an approximation of a truth values of $\mathbf{H}$-sentences, in terms of product topology.
  - First we introduce Canitini's model of $\mathbf{H}^{\Sigma_0}$, the quantifier-free fragment of $\mathbf{H}$, by using Tychonoff's fixed point theorem [Can08],
  - Next we demonstrait a typical way to extend his method to construct a model of $\mathbf{H}^{\Sigma_1}$, the Σ_1-fragment of $\mathbf{H}$.

Basic notions

- $\forall$Ł is defined by the standard $\mathbf{MV}$-algebra $[0,1]_{Ł}$:

(1) The truth values are real numbers in $[0,1]$,
(2) $||\bot|| = 0$, $||\varphi_0 \to \varphi_1|| = \min(1, 1 - ||\varphi_0|| + ||\varphi_1||)$,
(3) $||(\forall x)\varphi(x)|| = \inf\{||\varphi(a)||_{\mathbf{M}} : a \in |\mathbf{M}|\}$

- Let $\mathbf{H}$ be a set theory within $\forall$Ł
    - a binary predicate $\in$, and terms of the form $\{x : \varphi(x)\}$,
    - whose only axiom scheme is **the comprehension principle** (with parameters): For any φ not containing u freely,

$$(\forall u)[u \in \{x : \varphi(x, \cdots)\} \equiv \varphi(u, \cdots)]$$

- A sub-theory of $\mathbf{H}$:
    - $\mathbf{H}^{\Sigma_0}$ = CP is restricted to open formulae,
    - $\mathbf{H}^{\Sigma_1}$ = CP is restricted to Σ_1 formulae.

History

- Skolem [Sko57]
  - he conjectured CP doesn't imply a contradiction within $\forall Ł$,
  - its consistency (w/o any parameters) is showed by compactness and Brower's fixed point theorem [Cha63],
- Chang [Cha63], Fenstad [Fen64]
  - they proved the consistency of CP (restricted to formulae with parameters in a few special forms),
  - however they could not prove the whole consistency of full $\mathbf{H}$ because they failed to construct a model of full $\mathbf{H}$,
- White [Whi79].
  - he proved the consistency of $\mathbf{H}$ in a proof theoretic way,
  - very long and complex proof, no models.

What is a difficulty of constructing a model of $\mathbf{H}^{\Sigma_0}$?

- It's due to the circular nature of $\mathbf{H}^{\Sigma_0}$; **any formula has a two ways to calculate its truth value**.
- **Example:** let $\mathbf{M}$ is a natural Tarskian model of **H**, then the following must never happen. For any term t and any formula $\varphi_0(s) \wedge \varphi_1(s)$,

calculated-by-name (top-down):

$$\mathbf{M}(s \in \{x : \varphi_0(x) \wedge \varphi_1(x)\}) = 1$$

calculated-by-value (botom-up):

$$\mathbf{M}(\varphi_0(x)) = 0 \ \& \ \mathbf{M}(\varphi_1(x)) = 0.5$$

then $||\varphi_0(s) \wedge \varphi_1(s)||_{\mathbf{M}} = 0 \times ||\varphi_1(s)||_{\mathbf{M}} = 0$

- Two values are inconsistent: some special technique (as fixed point operation, etc.) are needed to coincide them.

cbn and cbv value

- $\mathbf{cbn}$-value (**calculated-by-name**):
  - $\mathbf{cbn}(\varphi(s)) = r$ iff $s \in \{x : \varphi(x)\}$'s truth value is assigned to be r,
- $\mathbf{cbv}$-value (**calculated-by-value**):
  - $\mathbf{cbv}(\varphi(s)) = r$ iff $\varphi(x)$ does not contain any logical connectives and $\mathbf{cbn}(\varphi(s)) = r$
  - $\mathbf{cbv}(\varphi \to \psi) = \min\{1 - \mathbf{cbv}(\varphi) + \mathbf{cbv}(\psi), 1\}$,
  - $\mathbf{cbv}(\bot) = 0$.

  They are **continuous**.
- If the assignment is adequate, $\mathbf{cbn}(\varphi) = \mathbf{cbv}(\varphi)$ for any φ.

- **Example:**
  - Let us assign $\mathbf{cbn}(R \in R) = 0.7$ where $R = \{x : x \notin x\}$,
  - then $\mathbf{cbv}(R \notin R) = 1 - \mathbf{cbn}(R \in R) = 0.3$,
  - therefore $\mathbf{cbv}(R \in R) = \mathbf{cbv}(R \notin R) = 0.3$,

Defining a truth function F

- Let us enumerate all formulae of $\mathbf{H}^{\Sigma_0}$: $\langle \varphi_n : n \in \omega \rangle$,
- Let us enumerate **cbn**, **cbv**-values of formulae:
  - $\langle r_n \in [0,1] : \mathbf{cbn}(\varphi_n) = r_n,\ n \in \omega \rangle \in [0,1]^\omega$,
  - $\langle r'_n \in [0,1] : \mathbf{cbv}(\varphi_n) = r'_n,\ n \in \omega \rangle \in [0,1]^\omega$,
- Let us define a function $F : [0,1]^\omega \to [0,1]^\omega$ by:

$$F(\langle r_n : \mathbf{cbn}(\varphi_n) = r_n \rangle) = \langle r'_n : \mathbf{cbv}(\varphi_n) = r'_n \rangle$$

- **Example:**
  - Let us assume $F' : [0,1] \times [0,1] \to [0,1] \times [0,1]$,
    - assume the assignment $\mathbf{cbn}(R \in R) = 0.7$, $\mathbf{cbn}(R \notin R) = 0.4$,
    - then $\mathbf{cbv}(R \notin R) = 0.3$, $\mathbf{cbv}(R \in R) = 0.3$,
    - Then $F'((0.7, 0.4)) = (0.3, 0.3)$.
  - The fixed point of F' is only $(0.5, 0.5)$:

$$F'((0.5, 0.5)) = (0.5, 0.5)$$

Cantini's model of $\mathbf{H}^{\Sigma_0}$

- Let us make it clear:
  - $\mathbb{R}^\omega$ is a (infinite-dimensional) locally convex topological vector space,
  - $[0, 1]^\omega$ is a compact, convex set,
  - $F : [0, 1]^\omega \to [0, 1]^\omega$ is continuous,
- There is a **fixed point** of F, $r \in [0, 1]^\omega$ s.t. $F(r) = r$!
  - by Tychonoff's theorem,
  - this r is a **model** of $\mathbf{H}^{\Sigma_0}$: $\mathbf{cbn}$ value and $\mathbf{cbv}$ value are coincide for any φ_n (it is r_n) [Can08].

The problem of constructing a model of $\mathbf{H}^{\Sigma_1}$

- **Next goal:** To construct a model of $\mathbf{H}^{\Sigma_1}$.
- **Problem:**
  - To handle with two values, $\mathbf{cbn}$ and $\mathbf{cbv}$, we need a fixed point argument.
  - Tychonoff's fixed point theorem needs that truth functions are continuous.
  - However, since $||\exists x\varphi(x)|| = \mathbf{sup}\{||\varphi(a)|| : a \in A\}$, taking sup violates continuity!
- **Solution:** To approximate the value of $||\exists x\varphi(x)||$ by finite domain, i.e. $||\exists x\varphi(x)||$ is approximated by $\mathbf{max}\{||\varphi(s)|| : s \in D\}$ for some finite D.
  - Forcing construction is essentially based on the same principle.

n-approximation

- $\mathbf{cbn}^n$-value: $\mathbf{cbn}^n : \mathbf{Form} \to [0,1]$
  - $\mathbf{cbn}^n(s \in \{x : \varphi(x)\}) = r$
- Let us enumerate all formulae of $\mathbf{H}^{\Sigma_1}$: $\langle \varphi_n : n \in \omega \rangle$,
  - **appear**$(\psi) = n$ s.t. ψ is φ_n in the list,
- $\mathbf{cbv}^n$-value: $\mathbf{cbv}^n : \mathbf{Form} \to ([0,1]^\omega \to [0,1])$
  - $\mathbf{cbv}^n(\varphi(s))(x) = r$
    where $\varphi(y)$ does not contain any logical connectives and $\mathbf{cbn}^n(\varphi(s)) = r$ where r is i-th coordinate of r and φ_i is $\varphi(s)$ in the list,
  - $\mathbf{cbv}^n(\varphi \to \psi)(x) = \min\{1 - r_0 + r_1, 1 : r_0 = \mathbf{cbv}^n(\varphi) \ \wedge \ r_1 = \mathbf{cbv}^n(\psi)\}$
  - $\mathbf{cbv}^n(\bot)(x) = \{0\}$,
  - $\mathbf{cbv}^n(\exists x \varphi(x))(x) = \max\{r_i : r_i = \mathbf{cbn}^n(\varphi(s))$ for $\mathbf{appear}(\varphi(s)) < n\}$
    This is a maximam of first (at most) n-many $\mathbf{cbn}(\varphi(s))$ in the $x \in [0,1]^\omega$

Defining F_n, a n-approximation of a truth function

- Let us enumerate $\mathbf{cbn}^n$, $\mathbf{cbv}^n$-values of formulae:
  - $\langle r_n \in [0,1] : \mathbf{cbn}^n(\varphi_n) = r_n,\ n \in \omega\rangle \in [0,1]^\omega$,
  - $\langle r'_n \in [0,1] : \mathbf{cbv}^n(\varphi_n) = r'_n,\ n \in \omega\rangle \in [0,1]^\omega$,
- Let us define a function $F^n : [0,1]^\omega \to [0,1]^\omega$ by:

$$F(\langle r_n : \mathbf{cbn}^n(\varphi_n) = r_n\rangle) = \langle r'_n : \mathbf{cbv}^n(\varphi_n) = r'_n\rangle$$

  Clearly it is a continuous function.
- The fixed points:
  - For any $n \in \omega$, there is a **fixed point** of F^n, $r \in [0,1]^\omega$ s.t. $F^n(r) = r$!
    - by Tychonoff's theorem,
  - Therefore we have a sequence $\langle r^i : r^i = F^i(r_i) \& i \in \omega\rangle$.
  - The sequence is a that of approximation of truth values of models of $\mathbf{H}^{\Sigma_1}$ by finite domain (and the domain is getting wider).

Compactness

- $\langle r^i : i \in \omega \rangle$ contains a convergence sequence in $[0,1]^\omega$ b/c of compactness of $[0,1]^\omega$,
  - therefore we assume $\langle x_n \rangle$ itself is a convergent sequence wlog,
- let us define their limit point:

$$r = \lim_{i \in \omega} r^i$$

- Any forcing constructions, versions of fixed point constructions are done by this way;

getting fixed points first, and taking limit of them second (thanks to compactness).

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