

# *Modal MTL-algebras*

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## MTL-algebras

### Definition

An **MTL-algebra** is an algebra of the form  $\langle A, \circ, \rightarrow, \wedge, \vee, 0, 1 \rangle$  where  $\langle A, \wedge, \vee, 0, 1 \rangle$  is a bounded lattice (with 1 the greatest and 0 the least elements respectively) and  $\circ$  is a binary operation that is associative, commutative, has identity 1 and is residuated with  $\rightarrow$ , i.e.,

$$(\forall x, y, z \in A)(x \circ z \leq y \iff z \leq x \rightarrow y)$$

and satisfying the prelinearity identity

$$(\forall x, y)((x \rightarrow y) \vee (y \rightarrow x) = 1)$$

The variety of MTL-algebras are the equivalent algebraic semantics for the monoidal t-norm logic (MTL for short), as defined in (Esteva and Godo [3]).

## Modal MTL-algebras

### Definition

A **modal MTL-algebra** is an algebra of the form  $\langle A, \circ, \rightarrow, \vee, \wedge, f, 0, 1 \rangle$ , where  $\langle A, \circ, \rightarrow, \vee, \wedge, 0, 1 \rangle$  is an MTL-algebra and  $f$  is a unary operation which is order preserving, i.e.,  $x \leq y \Rightarrow f(x) \leq f(y)$ .

### Definition

A **modal MTL-chain** is a modal MTL-algebra whose underlying lattice order is linearly ordered.

## Example of an MTL-chain

### Example

$\langle [0, 1], \circ, \rightarrow, \vee, \wedge, f, 0, 1 \rangle$  where

1.  $f = \Delta$  is called the Baaz Delta which is defined by: for any element  $e$

$$\begin{aligned}\Delta e &= 1 \text{ if } e = 1 \\ &= 0 \text{ otherwise.}\end{aligned}$$

2.  $f = !$  is called a storage operator (or linear logic exponential) and assigns to any element  $a \in [0, 1]$  the greatest idempotent below  $a$ .

In (Ciabattoni et al. [1]) and (Montagna [7]) operators on MTL related to  $!$  are considered.

## Filters of modal MTL-algebras

Let  $\mathbf{A} = \langle A, \circ, \rightarrow, \wedge, \vee, f, 0, 1 \rangle$  be a fixed modal MTL-algebra throughout this section.

### Definition

A nonempty subset  $F$  of  $A$  is a **filter** (often called an ideal) of  $\mathbf{A}$  if the following conditions are met:

1.  $1 \in F$ .
2. If  $a, b \in F$  then  $a \circ b \in F$ .
3. If  $a \in F$  and  $a \leq b$  then  $b \in F$ .
4. If  $a \in F$  and  $c \in A$  then  $f(c) \rightarrow f(c \circ a) \in F$ .

# A 1 – 1 correspondence between congruences and filters

## Lemma

If  $\theta$  is a congruence on  $\mathbf{A}$  then  $[1]_\theta$  is a filter.

## Lemma

If  $F$  is a filter of  $\mathbf{A}$  then  $\theta_F$  defined by

$\theta_F = \{(a, b) : a \rightarrow b, b \rightarrow a \in F\}$  is a congruence of  $\mathbf{A}$

## Lemma

If  $F$  is a filter of  $\mathbf{A}$ , then  $[1]_{\theta_F} = F$ .

## Corollary

The congruence lattice,  $\text{Con } \mathbf{A}$  (ordered by inclusion) is isomorphic to the filter lattice,  $\text{Filter } \mathbf{A}$ , (also ordered by inclusion), i.e.,

$$\text{Con } \mathbf{A} \cong \text{Filter } \mathbf{A}$$

## Axiomatizing modal MTL-algebras

### Proposition

The variety of modal MTL-algebras generated by the modal MTL-chains is axiomatized, relative to MTL, by

$$\forall x \forall y \forall z (x \vee z = 1 \Rightarrow (f(y) \rightarrow f(x \circ y)) \vee z = 1)$$

or

$$\forall x \forall y \forall z ([f(y) \rightarrow f((x \rightarrow z) \circ y)] \vee (z \rightarrow x) = 1)$$

## Stable subsets

We recall some definitions and results required for the MacNeille completion [6].

Let  $\mathbf{A} = \langle A, \circ, \rightarrow, \vee, \wedge, f, 0, 1 \rangle$  be a fixed modal MTL-chain.

### Definition

For  $K \subseteq A$ , define

- ▶  $K^\ell = \{a \in A \mid (\forall b \in K) a \leq b\}$
- ▶  $K^u = \{a \in A \mid (\forall b \in K) b \leq a\}$

A set  $X \subseteq A$  is **stable** if  $X^{\ell u} = X$ . Let  $F$  denote the sets of all stable sets of  $A$ .

## A complete lattice order on the stable sets

### Lemma

The order  $\supseteq$  is a complete lattice order on the set,  $F$ , of stable subsets of  $A$ .

The lattice operations on  $F$  associated with the lattice ordering  $\supseteq$  are:

1.

$$\bigvee_{i \in I}^F X_i = \bigcap_{i \in I} X_i$$

2.

$$\bigwedge_{i \in I}^F X_i = \bigcap \{ Y \in F \mid (\forall i \in I) X_i \subseteq Y \}$$

## Extending the operations to the complete structure

### Definition

For  $X, Y \in F$  define  $\circ^F$  and  $\rightarrow^F$  in  $\mathbf{F}$  by:

- ▶  $X \circ^F Y = (X^\ell \circ Y^\ell)^u$ ;
- ▶  $X \rightarrow^F Y = \{a \in A \mid (X^\ell \circ \{a\})^u \supseteq Y\}^u$

### Definition

For  $X \in F$  define  $f^F$  in  $\mathbf{F}$  by:

$$f^F(X) = \left\{ f(x) \mid x \in X^\ell \right\}^u$$

## Order preserving

On an MTL-chain, the order-preserving assumption,

$$(\forall x, y \in A) \quad x \leq y \Rightarrow f(x) \leq f(y),$$

on a modality  $f$  is equivalent to either of the identities:

$$f(x \vee y) = f(x) \vee f(y) \text{ or } f(x \wedge y) = f(x) \wedge f(y).$$

### Lemma

The operation  $f^{\mathbf{F}}$  is order preserving.

### Equivalently:

$f^{\mathbf{F}}$  distributes over joins (meets).

## The completion

### Theorem

The algebra  $\mathbf{F} = \langle F, \circ^{\mathbf{F}}, \rightarrow^{\mathbf{F}}, \vee^{\mathbf{F}}, \wedge^{\mathbf{F}}, f^{\mathbf{F}}, 0^{\mathbf{F}}, 1^{\mathbf{F}} \rangle$  is a complete modal MTL-chain and there exists an embedding of  $\mathbf{A}$  into  $\mathbf{F}$  that preserves all existing meets and joins in  $\mathbf{A}$ .

The map  $\iota : A \rightarrow F$  defined by  $\iota(a) = \{a\}^u$  is an embedding which preserves all existing meets and joins in  $\mathbf{A}$ .

The newly added modality is also preserved by this embedding.

## Proving preservation of properties – I

Given a term  $t$ , we wish to define an approximating term,  $t^\exists$ , in order to employ the methods used by Jonsson in [5] to prove preservation of properties (eg. identities, quasi-identities or inequalities).

### Definition

A term  $t(\vec{X})$  is called

- ▶ stable if  $t^F(\vec{X}) = t^\exists(\vec{X})$ .
- ▶ expanding if  $t^F(\vec{X}) \geq t^\exists(\vec{X})$ .
- ▶ contracting if  $t^F(\vec{X}) \leq t^\exists(\vec{X})$  ( $\forall \vec{X} \in F$ )

## Proving preservation of properties – II

To show that an inequality  $t(\vec{x}) \leq s(\vec{x})$  is satisfied in  $\mathbf{F}$  we should show that

- ▶  $t(\vec{X})$  is contracting,
- ▶  $s(\vec{X})$  is expanding, and that
- ▶  $\mathbf{A} \models t(\vec{X}) \leq s(\vec{X})$  implies that  $t^\exists(\vec{X}) \leq s^\exists(\vec{X})$

Then

$$t^{\mathbf{F}}(\vec{X}) \leq t^\exists(\vec{X}) \leq s^\exists(\vec{X}) \leq s^{\mathbf{F}}(\vec{X})$$

## The approximation

### Definition

For each term  $t(\vec{x})$  and  $\vec{X} = X_1, X_2, \dots, X_n \in F$  we define

$$t^{\exists}(\vec{X}) = \{t(\vec{a}) : \vec{a} \in X_1^{\ell}, X_2^{\ell}, \dots, X_n^{\ell}\}^u$$

### Example

Let  $t(x, y) = f(x \circ y)$  and  $X, Y \in F$ . Then

$$t^{\mathbf{F}}(X, Y) = \{f(a) : a \in (X \circ^{\mathbf{F}} Y)^{\ell}\}^u$$

$$t^{\exists}(X, Y) = \{f(a \circ b) : a \in X^{\ell}, b \in Y^{\ell}\}^u$$

## Positive and negative terms

### Definition

The sets of *positive* and *negative* terms are the smallest sets of terms closed under the following rules:

1. 0 and 1 are both positive and negative;
2. the term  $t(x) = x$  is positive for each variable  $x$ ;
3. if  $s$  is negative and  $t$  is positive then  $s \rightarrow t$  is positive and  $t \rightarrow s$  is negative;
4. if  $s(x_1, \dots, x_n)$  is a  $\{\circ, \wedge, \vee, f\}$  term and  $t_i$  for  $i = 1, \dots, n$  are positive (resp. negative) terms, then  $s(t_1, \dots, t_n)$  is positive (resp. negative).

## Identifying stable, contracting or expanding terms

### Lemma

The term  $t(x) = f(x)$  is a stable term.

### Proposition

1. Every positive term is expanding.
2. Every negative term is contracting.

### Proposition

If  $s_1$  and  $s_2$  are stable (resp. contracting, expanding) terms, then  $s_1 \vee s_2$  is stable (resp. contracting, expanding).

### Proposition

Any finite meet of terms of the form  $f(x)$  is stable.

## Additional axioms preserved

### Proposition

If  $\mathbf{A}$  satisfies any of the following axioms involving  $f$ , then so does  $\mathbf{F}$

1.  $f(0) = 0$
2.  $f(1) = 1$
3.  $f(x) \leq x$
4.  $f(x) \circ f(x) = f(x)$
5.  $f(x) \vee (f(x) \rightarrow 0) = 1$
6.  $f(x \rightarrow y) \leq f(x) \rightarrow f(y)$

### Lemma

If  $f(x) \leq x$  and  $f(f(x)) = f(x)$ , then  $f^{\mathbf{F}}(f^{\mathbf{F}}(X)) = f^{\mathbf{F}}(X)$

## Box-like operations

We will call an operator  $f$  ‘box-like’ if it satisfies the identity

$$f(x) \circ f(y) \leq f(x \circ y)$$

### Motivation:

In a modal algebra an operation is ‘box-like’ if the identity

$$f(x \wedge y) = f(x) \wedge f(y) \tag{1}$$

is satisfied. However, since  $\circ$  and  $\wedge$  coincide in a modal algebra (1) is satisfied if and only if the inequality

$$f(x \rightarrow y) \leq f(x) \rightarrow f(y) \tag{2}$$

is satisfied.

In a modal MTL-algebra (2) is satisfied if and only if

$$f(x) \circ f(y) \leq f(x \circ y)$$

is satisfied.

## Preservation theorems for box-like operations

### Lemma

If  $f$  is box-like in  $\mathbf{A}$  then so is  $f^{\mathbf{F}}$  in  $\mathbf{F}$ .

### Proposition

If  $\alpha$  is a term

- ▶ built up from boxed variables and negative terms using only  $\nabla$ ; or
- ▶ built from boxed variables using only  $\wedge$

and  $\beta$  is a positive term, then the equality  $\alpha \rightarrow \beta = 1$  is preserved by the completion.

The is a ‘Sahlqvist-like’ syntactic descriptions of identities preserved.

## A Left-continuous operation

### Lemma

If  $f$  is left-continuous, then so is  $f^F$  i.e., the equality

$f(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} (f(x_i))$  (whenever  $\bigvee_{i \in I} x_i$  exists) is preserved by the completion.

## Preservation theorems for left-continuous operations

### Lemma

If  $f$  is left-continuous and  $s$  is an expanding (resp. contracting, stable) term, then  $f(s)$  is also an expanding (resp. contracting, stable) term.

### Proposition

If  $\alpha$  is built up from negative terms using only  $\vee$  and  $f$  and  $\beta$  is a positive term, then the equality  $\alpha \rightarrow \beta = 1$  is preserved by the completion.

## Adding an order-reversing modality

### Example

$\langle [0, 1], \circ, \rightarrow, \vee, \wedge, \sim, 0, 1 \rangle$  where  $\sim$  is defined by:  
for any element  $e \in [0, 1]$

$$\sim e = 1 - e$$

Let  $\mathbf{A} = \langle \mathbf{A}, \circ, \rightarrow, \vee, \wedge, f, \sim, 0, 1 \rangle$  be a MTL-algebra with an additional unary operation,  $\sim$ , which is order-reversing.

### Definition

For  $X \in \mathbf{F}$  define  $\sim^{\mathbf{F}}$  in  $\mathbf{F}$  as follows

$$\sim^{\mathbf{F}} X = \{ \sim a : a \in X \}^u$$

## Properties preserved involving the order-reversing modality

### Lemma

The order-reversing operation  $\sim^{\mathbf{F}}$  is preserved by the embedding  $\iota : \mathbf{A} \rightarrow \mathbf{F}$  of  $\mathbf{A}$  into  $\mathbf{F}$ .

### Lemma

The operation  $\sim^{\mathbf{F}}$  is order-reversing.

### Corollary

$\mathbf{F}$  satisfies the identity  $\sim (X \vee Y) = \sim X \wedge \sim Y$

### Lemma

If  $\mathbf{A}$  satisfies  $\sim 0 = 1$  or  $\sim \sim x = x$ , respectively, then so does  $\mathbf{F}$ .

### Lemma

Let  $t$  be an expanding term, then  $\sim t$  is contracting.

Since the term  $x$  is stable, the term  $t(x) = \sim x$  is contracting.

## Partial subalgebras

### Definition

Given a modal MTL-algebra  $\mathbf{A} = \langle A, \circ, \rightarrow, \vee, \wedge, f, 0, 1 \rangle$  and any  $B \subseteq A$ , the **partial subalgebra**  $\mathbf{B}$  of  $\mathbf{A}$  with domain  $B$  is the partial algebra  $\langle B, \circ^{\mathbf{B}}, \rightarrow^{\mathbf{B}}, \wedge^{\mathbf{B}}, \vee^{\mathbf{B}}, f^{\mathbf{B}}, 0, 1 \rangle$  with, for  $b_1, b_2 \in B$

$$b_1 \circ^{\mathbf{B}} b_2 = \begin{cases} b_1 \circ b_2 & \text{if } b_1 \circ b_2 \in B \\ \text{undefined} & \text{if } b_1 \circ b_2 \notin B \end{cases}$$

$$b_1 \rightarrow^{\mathbf{B}} b_2 = \begin{cases} b_1 \rightarrow b_2 & \text{if } b_1 \rightarrow b_2 \in B \\ \text{undefined} & \text{if } b_1 \rightarrow b_2 \notin B \end{cases}$$

$$f^{\mathbf{B}}(b_1) = \begin{cases} f(b_1) & \text{if } f(b_1) \in B \\ \text{undefined} & \text{if } f(b_1) \notin B \end{cases}$$

$$b_1 \wedge^{\mathbf{B}} b_2 = b_1 \wedge b_2 \quad b_1 \vee^{\mathbf{B}} b_2 = b_1 \vee b_2$$

## The finite embeddability property

### Definition

A class  $\mathcal{K}$  of algebras has the **finite embeddability property (FEP, for short)** if every finite partial subalgebra of some member of  $\mathcal{K}$  can be embedded into some finite member of  $\mathcal{K}$ .

### Applying the FEP:

If a class  $\mathcal{K}$  of algebras has the FEP then every universal sentence that fails in  $\mathcal{K}$  will fail in a finite member of  $\mathcal{K}$ . Thus, if  $\mathcal{K}$  is finitely axiomatized, then its universal theory is decidable.

## Subsets of $A$ used in the construction

### Goal:

To show that the class of modal MTL-chains has the FEP. To do so we must find a finite modal MTL-chain  $\mathbf{C}$  into which  $\mathbf{B}$  can be embedded.

Let  $\mathbf{A} = \langle A, \circ, \rightarrow, \vee, \wedge, f, 0, 1 \rangle$  be a fixed modal MTL-chain and  $\mathbf{B} = \langle B, \circ^{\mathbf{B}}, \rightarrow^{\mathbf{B}}, \vee^{\mathbf{B}}, \wedge^{\mathbf{B}}, f^{\mathbf{B}}, 0, 1 \rangle$  a fixed finite partial subalgebra of  $\mathbf{A}$ .

### Definition

Define the following subsets of  $A$ :

- ▶  $M$  is the  $\{\circ\}$ -closure of  $B$ .
- ▶  $M^* = \{a \rightarrow b : a \in M, b \in B\}$

## The set of stable elements in $M^*$

### Definition

For each  $a \in A$  define:

$$a^\ell = \max\{b \in M : b \leq a\}$$

$$a^u = \min\{c \in M^* : a \leq c\}$$

An element  $c \in M^*$  is **stable** if  $c = c^{\ell u}$ .

Let  $C$  denote the set of stable elements.

### Lemma

$\langle M, \leq \rangle$  is reverse well-ordered and  $\langle M^*, \leq \rangle$  is well-ordered.

### Proposition

The set  $C$  of stable elements is finite.

## The set of stable elements in $M^*$

If  $C$  is infinite it must contain at least one of

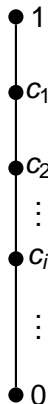
- ▶ an infinite descending chain;
- ▶ an infinite ascending chain.

## The set of stable elements in $M^*$

Suppose  $C$  contains an infinite descending chain  $c_1 > c_2 > \dots$ :

$C \subseteq M^*$  implies  $M^*$  contains the infinite descending chain  $c_1 > c_2 > \dots$

Contradiction -  $M^*$  is well-ordered.



## The set of stable elements in $M^*$

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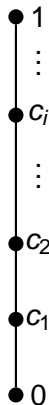
## The set of stable elements in $M^*$

Suppose  $C$  contains an infinite ascending chain  $c_1 < c_2 < \dots$ :

Then  $c_1^l < c_2^l < \dots$  is an infinite ascending chain in  $M$ :

If  $c_i^l = c_{i+1}^l$  then  $c_i = c_i^{lu} = c_{i+1}^{lu} = c_{i+1}$  since  $c_i, c_{i+1}$  are both stable - contradiction the fact that  $c_1, c_2, \dots$  is strictly ascending.

**Contradiction** -  $M$  is reverse well-ordered.



## The set of stable elements in $M^*$

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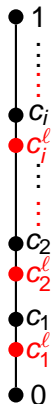
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**Contradiction** -  $M$  is reverse well-ordered.



## Extending the operations to $\mathbf{C}$

*Definition (van Alten [9])*

For  $c, d \in \mathbf{C}$ , define  $c \circ^{\mathbf{C}} d = (c^{\ell} \circ d^{\ell})^u$  and  
 $c \rightarrow^{\mathbf{C}} d = (c^{\ell} \rightarrow d)^{\ell u}$

*Definition*

For  $c \in \mathbf{C}$  define  $f^{\mathbf{C}}(c) = f(c^{\ell})^{\ell u}$ .

*Lemma*

If  $f$  is order-preserving then the operation  $f^{\mathbf{C}}$  is order-preserving.

Recall that on MTL-chains this implies that  $f^{\mathbf{C}}$  distributes over meets and joins.

## Modal MTL-chains has the FEP

### Theorem

The algebra  $\mathbf{C} = \langle C, \circ^{\mathbf{C}}, \rightarrow^{\mathbf{C}}, \wedge, \vee, f^{\mathbf{C}}, 0, 1 \rangle$  is a modal MTL-chain and there exists an embedding of  $\mathbf{B}$  into  $\mathbf{C}$ .

The map  $\iota : B \rightarrow C$  defined by  $\iota(b) = b$  is an embedding which preserves all the operations defined on  $B$ .

### Corollary

The class of modal MTL-chains has the FEP and hence, so does the variety of modal MTL-algebras generated by it.

## The objective

### Goal:

We wish to identify properties (eg. identities, quasi-identities or inequalities) preserved by the FEP construction.

We will define an approximating term  $t^*$  for each term  $t$  and then make use of methods similar to Jonsson's [5] to prove the preservation of properties.

## Our method for proving preservation of properties

### Definition

A term  $s(\vec{x})$  is called

- ▶  $*$ -stable if  $s^{\mathbf{C}}(\vec{c}) = s^*(\vec{c})$
- ▶  $*$ -expanding if  $s^{\mathbf{C}}(\vec{c}) \geq s^*(\vec{c})$
- ▶  $*$ -contracting if  $s^{\mathbf{C}}(\vec{c}) \leq s^*(\vec{c})$

To prove that  $t(\vec{c}) \leq s(\vec{c})$  is satisfied in  $\mathbf{C}$  we show that:

$t(\vec{c})$  is contracting,

$s(\vec{c})$  is expanding, and that

$\mathbf{A} \models t(\vec{c}) \leq s(\vec{c})$  implies that  $t^*(\vec{c}) \leq s^*(\vec{c})$ .

Then

$$t^{\mathbf{C}}(\vec{c}) \leq t^*(\vec{c}) \leq s^*(\vec{c}) \leq s^{\mathbf{C}}(\vec{c})$$

## The approximation

### Definition

For each term  $s(\vec{x})$  and  $\vec{c} \in \mathbf{C}$ , define

$$s^*(\vec{c}) = s(\vec{c}^\ell)^{\ell u}$$

### Example

Let  $s(x, y) = f(x \circ y)$  and  $c, d \in \mathbf{C}$ . Then

$$s^{\mathbf{C}}(c, d) = f((c^\ell \circ d^\ell)^{u\ell})^{\ell u}$$

$$s^*(c, d) = f(c^\ell \circ d^\ell)^{\ell u}$$

## Preservation of properties

### Lemma

If  $s(\vec{x})$  is a  $\{0, \vee, \wedge, 0, 1\}$ -term and  $\vec{c} \in \mathbf{C}$ , then  $f^{\mathbf{C}}(s^{\mathbf{C}}(\vec{c}))$  is  $*$ -expanding.

### Lemma

If  $s(\vec{x})$  is a  $\{\vee, \wedge\}$ -term, then  $t(\vec{c}) = f(s(\vec{c}))$  is  $*$ -stable.

### Lemma

Let  $s(x) = f(x \rightarrow 0)$ , then  $s(x)$  is a  $*$ -contracting term.

### Lemma

If  $\mathbf{A}$  satisfies either of  $f(x) \vee (f(x) \rightarrow 0) = 1$  or  $f(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} f(x_i)$  then so will  $\mathbf{C}$ .

## Modifications to the construction

In order for more properties to be preserved, we need to modify the construction in the following way:

### Definition

Redefine

$$M^f \text{ is the } \{\circ, f\}\text{-closure of } B.$$

$$M^* = \{a \rightarrow b : a \in M^f, b \in B\}$$

For  $\mathbf{C}$  to be finite we need to confirm that  $M$  remains reversed well-ordered and  $M^*$  is well-ordered.

## A decreasing operation

### Lemma

If  $f$  is decreasing, i.e.,  $f(a) \leq a$  for every  $a \in A$ , then the set  $M^f$  – the  $\{\circ, f\}$ -closure of  $B$  – is reversed well-ordered and  $M^*$  is well-ordered.

$M^f$ 's reversed well-orderedness follows directly Higman's Theorem. The fact that  $M^*$  is well ordered is a consequence of the analogous result for MLT-chains in [9].

### Lemma

If  $\mathbf{A}$  satisfies  $f(a) \leq a$ , then so does  $\mathbf{C}$ .

## An idempotent operation

If  $f$  is idempotent – i.e.,  $f(f(a)) = f(a)$  for every  $a \in A$  – it does not follow immediately from Higman's theorem that  $M^f$  is reversed well-ordered.

### Theorem

If  $f$  is idempotent, i.e.,  $f(f(x)) = f(x)$  for every  $x$ , then the set  $M^f$  – the  $\{\circ, f\}$ -closure of  $B$  – is reversed well-ordered and  $M^*$  is well-ordered.

### Lemma

If  $\mathbf{A}$  satisfies  $f(f(a)) = f(a)$ , then so does  $\mathbf{C}$ .

## Additional properties preserved under the modified construction

### Lemma

If  $M$  is closed under  $f$  and  $\mathbf{A}$  satisfies  $f(a) \circ f(a) = f(a)$  then so does  $\mathbf{C}$ .

### Lemma

If  $s$  is a  $\{\circ, \wedge, \vee, f, 0, 1\}$ -term then  $s$  is  $*$ -expanding.

### Lemma




If  $t = s \rightarrow 0$  and  $s$  is a  $\{\circ, \wedge, \vee, f, 0, 1\}$ -term then  $t$  is  $*$ -contracting.

# A Sahlqvist-like theorem for the modified FEP construction




## Theorem

If  $\alpha$  is built up from the negation of  $\{\circ, \wedge, \vee, f, 0, 1\}$ -terms using only  $\circ, \wedge$  and  $\vee$  and  $\beta$  is a  $\{\circ, \wedge, \vee, f, 0, 1\}$ -term then the identity  $\alpha \rightarrow \beta = 1$  is preserved by the modified FEP construction.




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